On the direct decomposition of nilpotent expanded groups

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Nilpotent groups

Theorem (classical result from group theory)

Let **G** be a finite nilpotent group. Then **G** is isomorphic to a direct product of groups of prime power order.

Sketch of the proof

Let S_p be a p-Sylow subgroup of **G**. Since **G** is nilpotent, $N_G(H) > H$ for all $H <$ **G**. By Sylow, $N_G(N_G(S_p)) \leq N_G(S_p)$, hence $N_G(S_p) = G$, and thus $S_p \trianglelefteq G$.

Description of nilpotent groups

Theorem (Characterisations of nilpotent groups)

Let $\mathbf{G}$ be a finite group, $k \in \mathbb{N}$. TFAE

1. $\mathbf{G}$ is nilpotent of class k
   $:\Leftrightarrow$ the lower central series $\gamma_1(\mathbf{G}) := G$, $\gamma_n(\mathbf{G}) := [G, \gamma_{n-1}(\mathbf{G})]$ satisfies $|\gamma_k(\mathbf{G})| > 1$, $|\gamma_{k+1}(\mathbf{G})| = 1$;
2. k is minimal in $\mathbb{N}$ with
   $$\exists p \in \mathbb{R}[x] : \deg(p) = k \text{ and } \forall n : |\mathbf{F}_{\mathcal{V}(\mathbf{G})}(n)| \leq 2^{p(n)};$$
3. the supremum of “the rank of commutator terms of $\mathbf{G}$” is k (see [Kearnes, 1999]);
4. $|[\![G, G, \ldots, G]\!]_k| > 1$ and $|[\![G, G, \ldots, G]\!]_{k+1}| = 1$ (see [Mudrinski, 2009]).

Nilpotence for expanded groups

Definition (Nilpotent expanded groups)

Let $\mathbf{V} = \langle V, +, -, 0, f_1, f_2, \ldots \rangle$ be an expanded group, $A, B \trianglelefteq \mathbf{V}$.

$$\llbracket A, B \rrbracket := \{p(a, b) \mid p \in \mathrm{Pol}_2(\mathbf{V}),$$
$$a \in A,\ b \in B,\ p(0,0) = p(a,0) = p(0,b) = 0\}.$$

$\mathbf{V}$ is *nilpotent of class k* if for $\gamma_1(\mathbf{V}) := V$, $\gamma_n(\mathbf{V}) := \llbracket V, \gamma_{n-1}(\mathbf{V}) \rrbracket$ we have $|\gamma_k(\mathbf{V})| > 1$, $|\gamma_{k+1}(\mathbf{V})| = 1$.

Remarks on $\llbracket \bullet, \bullet \rrbracket$

- In expanded groups, we consider *ideals = 0-classes of congruences* instead of congruences.
- $\llbracket A, B \rrbracket$ then corresponds to the *term-condition commutator* introduced in [Freese and McKenzie, 1987, McKenzie et al., 1987].

Example of a nilpotent expanded group

A nilpotent expansion of $\langle \mathbb{Z}_6, + \rangle$

Let $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ be defined by

x	$f(x)$
0	3
1	0
2	0
3	3
4	0
5	0.

Then $\mathbf{V}_6 := \langle \mathbb{Z}_6, +, -, 0, f \rangle$ is nilpotent of class 2, and its congruence lattice is a three element chain.

Facts on $\mathbf{V}_6$

Lemma

$\mathbf{V}_6$ is directly indecomposable, and $|\mathbf{F}_{\mathcal{V}(\mathbf{V}_6)}(n)| \geq 2^{2^n}$ for all $n \in \mathbb{N}$.

Kearnes's decomposition theorem

As a corollary of [Kearnes, 1999, Theorem 3.14] and [Hobby and McKenzie, 1988, Lemma 12.4], one obtains:

Theorem ([Kearnes, 1999])

Let **A** be a finite Mal'cev algebra such that $\exists p \in \mathbb{R}[x]$ with

$$|\mathbf{F}_{\mathcal{V}(\mathbf{A})}(n)| \leq 2^{p(n)} \text{ for all } n \in \mathbb{N}.$$

Then **A** is nilpotent and isomorphic to a direct product of algebras of prime power order.

Theorem ([Berman and Blok, 1987, Theorem 2])

Let **A** be finite, in a congruence modular variety, of finite type, nilpotent, direct product of algebras of prime power order. Then

$$\exists p \in \mathbb{R}[x] : |\mathbf{F}_{\mathcal{V}(\mathbf{A})}(n)| = 2^{p(n)} \text{ for all } n \in \mathbb{N}.$$

Absorbing polynomials and supernilpotence

Definition

$\mathbf{V} = \langle V, +, -, 0, f_1, f_2, \ldots \rangle$ expanded group, $p \in \mathrm{Pol}_n \mathbf{V}$. p is *absorbing* $:\Leftrightarrow \forall \mathbf{x} : 0 \in \{x_1, \ldots, x_n\} \Rightarrow p(x_1, \ldots, x_n) = 0$.

Definition (supernilpotent)

$\mathbf{V}$ expanded group, $k \in \mathbb{N}$. $\mathbf{V}$ is *supernilpotent of class* $k :\Leftrightarrow$

1. there is a nonconstant absorbing $p \in \mathrm{Pol}_k(\mathbf{V})$, and
2. $\forall\, n > k$ all n-ary absorbing polynomials are constant.

Characterisation of supernilpotent expanded groups

Lemma (Description of finite snp expanded groups)

Let **W** be a finite expanded group, $k \in \mathbb{N}$. TFAE

1. **W** is supernilpotent of class $k \in \mathbb{N}$;
2. k is minimal in $\mathbb{N}$ with

$$\exists p \in \mathbb{R}[x] : \deg(p) = k \text{ and } \forall n : |\mathbf{F}_{\mathcal{V}(\mathbf{W})}(n)| \leq 2^{p(n)};$$

3. the supremum of "the rank of commutator terms of **W**" is k (see [Kearnes, 1999]);
4. $|[\![W, W, \ldots, W]\!]_k| > 1$ and $|[\![W, W, \ldots, W]\!]_{k+1}| = 1$ (see [Mudrinski, 2009]).

Connections between nilpotent and supernilpotent

Lemma (Groups)

Let **G** be group. Then **G** is nilpotent of class $k \Leftrightarrow$ **G** is supernilpotent of class k.

Remark

$\Rightarrow$ requires commutator calculus; calculations done in [Aichinger and Ecker, 2006].

Lemma (Expanded groups)

A supernilpotent expanded group of class k is nilpotent of class $\leq k$.

Corollary of [Berman and Blok, 1987, Theorem 2]

A finite nilpotent expanded group of finite type and prime power order is supernilpotent.

Connections between nilpotent and supernilpotent

Theorem (EA, Mudrinski, 2011)

Let $k \geq 1$, $m \geq 2$, $\mathbf{V} = \langle V, +, -, 0, f_1, f_2, \dots \rangle$ expanded group such that all f_i are "multilinear" and of arity $\leq m$, and $\mathbf{V}$ is nilpotent of class k. Then $\mathbf{V}$ is supernilpotent of class $\leq m^{k-1}$.

Remark (the bound can be attained)

For all $k \geq 1$, $m \geq 2$, there is a finite nilpotent $\mathbf{V}$ of class k with all f_i "multilinear" and of arity $\leq m$ such that $\mathbf{V}$ is supernilpotent of class m^{k-1}.

Colouring the prime sections of the congruence lattice

Definition (Characteristic of a prime section)

Let $\mathbf{V}$ be an expanded group, and let $A \prec B \trianglelefteq \mathbf{V}$, $[\![B, B]\!] \leq A$. Then $\mathrm{char}(A, B)$ is the exponent of $\langle B/A, +\rangle$.

Remark

$R := \langle P_0(\mathbf{V})/Ann(B/A), +, \circ\rangle$ is a ring with simple module $M := B/A$. Hence $\mathrm{char}(A, B)$ is the characteristic of the division ring $\mathrm{End}_R(B/A)$.

Characteristic is prime or zero

Let $\mathbf{V}$ be an expanded group, and let $A \prec B \trianglelefteq \mathbf{V}$, $[\![B, B]\!] \leq A$. Then $\mathrm{char}(A, B) \in \mathbb{P} \cup \{0\}$.

Monochromatic expanded groups

Definition (A generalisation of "prime power order")

Let **V** be a solvable expanded group. **V** is *monochromatic* if all prime sections in the ideal lattice have the same colour.

Theorem (EA, 2012)

Let **V** be a supernilpotent expanded group whose ideal lattice is of finite height. Then **V** is isomorphic to a direct product of finitely many monochromatic expanded groups.

Proof of this decomposition result

Lemma

Let **R** be a ring with unit, and let **M** be a unitary **R**-module such that **M** has exactly three submodules; let Q be the submodule different from 0 and M. Then the exponents of the groups $\langle M/Q, +\rangle$ and $\langle Q, +\rangle$ are equal.

Lemma (cf. [Mayr, 2008, Lemma 3])

Let **V** be a finite expanded group whose ideal lattice is a three element chain $\{0\} < Q < V$. We assume that the exponents of the groups $\langle Q, +\rangle$ and $\langle V/Q, +\rangle$ are different, and that $[V, V] = Q$ and $[V, Q] = 0$. Then **V** is not supernilpotent.

Main tool in the proof

The operation of the polynomial ring

$$
\begin{aligned}
M &:= \{p \in \mathrm{Pol}_1 \mathbf{V} : p(V) \subseteq Q, \\
&\qquad p \text{ is constant on each } Q\text{-coset}\}, \\
R &:= \mathbb{Z}[t], \;\; w \in V, \\
r \star_w m(x) &:= \textstyle\sum_{i=0}^{\deg(r)} r_i * m(x + i * w) \;\text{ for } m \in M, x \in V.
\end{aligned}
$$

Use of this operation

- For all $m \in \mathbb{N}$, there is $w \in V$, $f \in M$ such that

$$(t-1)^m \star_w f \text{ is not constant.}$$

- From this, we will produce absorbing polynomials of arbitrary arity.

Produce absorbing polynomials of arbitrary arity

Task

Produce absorbing nonconstant polynomial of arity m.

Define a sequence

- Choose $f \in M$, $w \in W$ such that $(t-1)^{m-1} \star_w f$ is not constant.
- Define
  - $h^{(1)}(x_1) := f(x_1) - f(0)$.
  - $h^{(n)}(x_1, \ldots, x_n) :=$ $h^{(n-1)}(x_1 + x_n, x_2, \ldots, x_{n-1}) - h^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + h^{(n-1)}(0, x_2, \ldots, x_{n-1}) - h^{(n-1)}(x_n, x_2, \ldots, x_{n-1})$.
- Then $h^{(n)}(x_1, w, \ldots, w) =$ $((t-1)^{n-1} \star_w f)\,(x_1) - ((t-1)^{n-1} \star_w f)\,(0)$ for all $x_1 \in V$.

Aichinger, E. and Ecker, J. (2006).
Every $(k+1)$-affine complete nilpotent group of class k is affine complete.
Internat. J. Algebra Comput., 16(2):259–274.

Berman, J. and Blok, W. J. (1987).
Free spectra of nilpotent varieties.
Algebra Universalis, 24(3):279–282.

Freese, R. and McKenzie, R. N. (1987).
Commutator Theory for Congruence Modular varieties, volume 125 of *London Math. Soc. Lecture Note Ser.*
Cambridge University Press.

Hobby, D. and McKenzie, R. (1988).
The structure of finite algebras, volume 76 of *Contemporary mathematics*.
American Mathematical Society.

Kearnes, K. A. (1999).
Congruence modular varieties with small free spectra.

Algebra Universalis, 42(3):165–181.

Mayr, P. (2008).
Polynomial clones on squarefree groups.
Internat. J. Algebra Comput., 18(4):759–777.

McKenzie, R. N., McNulty, G. F., and Taylor, W. F. (1987).
Algebras, lattices, varieties, Volume I.
Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, California.

Mudrinski, N. (2009).
On Polynomials in Mal'cev Algebras.
PhD thesis, University of Novi Sad.