TECHNISCHE UNIVERSITÄT DRESDEN

DECOMPOSING DISTRIBUTIVE LATTICES UP TO POLYNOMIAL EQUIVALENCE USING RST

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Outline

1. Theoretical background from Relational Structure Theory

2. Polynomial expansions of distributive lattices

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2. Polynomial expansions of distributive lattices

Localising finite algebras

- (finite, nonempty) algebras $\mathbf{A} = \langle A; F \rangle$, where $F \subseteq O_A = \bigcup_{n \in \mathbb{N}} A^{A^n}$
- analysis up to term equivalence, i.e. equality of $\mathrm{Clo}(\mathbf{A}) = \mathrm{Term}(\mathbf{A})$
- restriction of algebras to subsets $U \subseteq A$
  $\mathbf{A}|_U := \left\langle U; \left\{ f\restriction^{U}_{U^{\operatorname{ar} f}} \;\middle|\; f \in \mathrm{Clo}(\mathbf{A}) \wedge f\left[U^{\operatorname{ar} f}\right] \subseteq U \right\} \right\rangle$
- in fact, not ordinary subsets,

Definition (neighbourhood)

$U \in \mathrm{Neigh}\,\mathbf{A} \;:\Longleftrightarrow\; U = e[A]$ for some
$e \in \mathrm{Idem}\,\mathbf{A} := \left\{ g \in \mathrm{Clo}^{(1)}(\mathbf{A}) \mid g \circ g = g \right\}$

Localising finite algebras via relations

- relational counterpart $\underset{\sim}{\mathbf{A}} = \langle A; \operatorname{Inv} \mathbf{A} \rangle$, where $\operatorname{Inv} \mathbf{A} := \bigcup_{m \in \mathbb{N}_+} \operatorname{Sub} \mathbf{A}^m$
- restriction to neighbourhoods $U \in \operatorname{Neigh} \mathbf{A}$ $\underset{\sim}{\mathbf{A}}\restriction_U := \langle U; \{ S\restriction_U \mid S \in \operatorname{Inv} \mathbf{A} \} \rangle$, where $S\restriction_U := S \cap U^m$
- corresponds to $\mathbf{A}|_U$.

Separating invariant relations

Definition (Separation, Cover)

Let $m \in \mathbb{N}_+$, $S, T \in \mathrm{Inv}^{(m)}\mathbf{A} := \mathrm{Sub}\,\mathbf{A}^m$,

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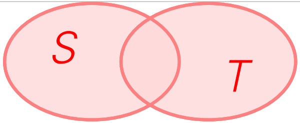

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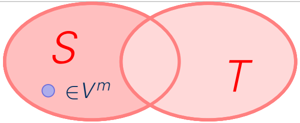

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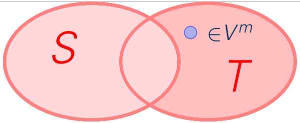

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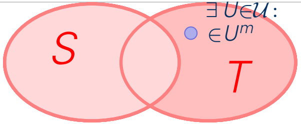

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Let $m \in \mathbb{N}_+$, $S, T \in \operatorname{Inv}^{(m)} \mathbf{A} := \operatorname{Sub} \mathbf{A}^m$, $V \in \operatorname{Neigh} \mathbf{A}$ and $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$.

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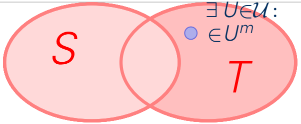

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- V separates S and T iff $S{\restriction_V} \neq T{\restriction_V}$. $\quad (S{\restriction_V} := S \cap V^m)$
- $\mathcal{U}$ separates S and T iff $\exists U \in \mathcal{U}$: U separates S and T (i.e. $S{\restriction_U} \neq T{\restriction_U}$)
- $\mathcal{U} \in \mathrm{Cov}(\mathbf{A})$ iff $\mathcal{U}$ covers $\mathbf{A}$ iff every nonidentical pair $S', T' \in \mathrm{Inv}^{(\ell)}\mathbf{A}$, $S' \neq T'$, $\ell \in \mathbb{N}_+$, is separated by $\mathcal{U}$:

$$S' \neq T' \quad \Longrightarrow \quad \exists U \in \mathcal{U}\colon S'{\restriction_U} \neq T'{\restriction_U}.$$

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Theorem (Kearnes, Á. Szendrei, 2001)

Let $\mathbf{A}$ *be a finite algebra and* $E \subseteq \operatorname{Idem} \mathbf{A}$*. Set* $\mathcal{U} := \{\operatorname{im} e \mid e \in E\}$*. T.f.a.e.:*

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2. $\exists q \in \mathbb{N}\, \exists e_1, \dots, e_q \in E\, \exists g_1, \dots, g_q \in \operatorname{Clo}^{(1)}(\mathbf{A})$ $\exists \lambda \in \operatorname{Clo}^{(q)}(\mathbf{A}) :\ \lambda \circ (e_1 \circ g_1, \dots, e_q \circ g_q) = \operatorname{id}_A$.

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3. $\exists q \in \mathbb{N}\, \exists (U_1, \ldots, U_q) \in \mathcal{U}^q :$ $\underset{\sim}{\mathbf{A}}$ *is a retract of* $\underset{\sim}{\mathbf{A}}\restriction_{U_1} \times \cdots \times \underset{\sim}{\mathbf{A}}\restriction_{U_q}$

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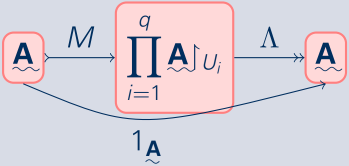

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3. $\mathrm{Var}\,\mathbf{A} \equiv \mathrm{Var}\left(\boxtimes_{i=1}^{q} \mathbf{A}|_{U_i}\right)$

When localisation is useless . . .

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$\mathbf{A}$ irreducible, iff every cover $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ contains $A \in \mathcal{U}$.

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Definition (Irreducible neighbourhood)

$U \in \operatorname{Neigh} \mathbf{A}$ irreducible $:\Longleftrightarrow$ $\mathbf{A}|_U$ irreducible.

$\operatorname{Irr}(\mathbf{A}) := \{\, U \in \operatorname{Neigh} \mathbf{A} \mid U \text{ is irreducible}\}.$

Better localisations–refinement of covers

For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(\mathbf{A})$:

$\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$ quasiorder

$:\Longleftrightarrow \forall V \in \mathcal{V} \exists U \in \mathcal{U} \colon V \subseteq U$

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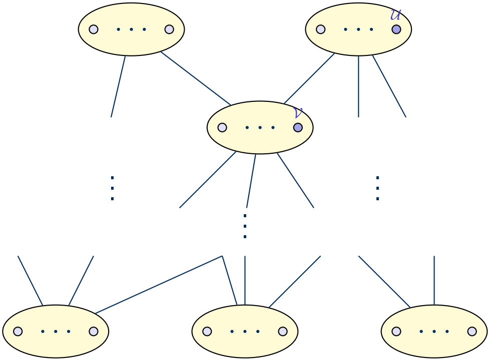

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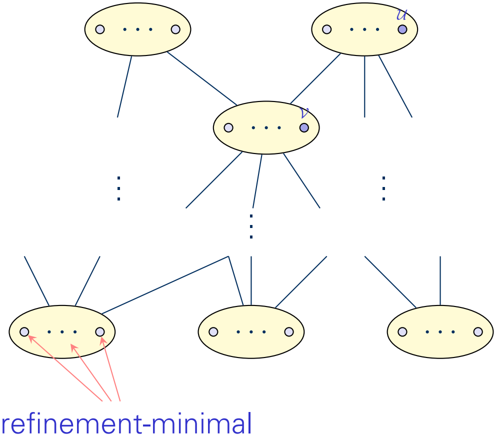

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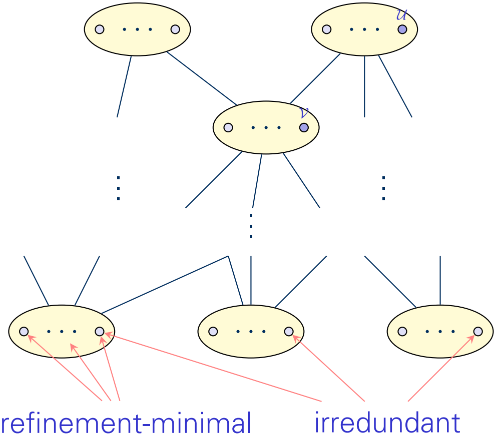

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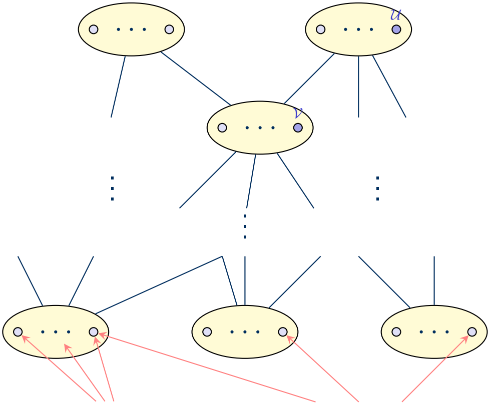

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refinement-minimal $\wedge$ irredundant $\Longleftrightarrow: \mathcal{U} \in \mathrm{Cov}(\mathbf{A})$ nonrefinable

Existence and uniqueness of covers

Theorem (Kearnes, Á. Szendrei, 2001, MB, 2009)

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Theorem (MB, FMS, 2011)

The unique nonrefinable cover of a finite algebra $\mathbf{A}$ *consists of a system of* $\cong$*-representatives of the* ***maximal strictly irreducible*** *neighbourhoods of* $\mathbf{A}$*.*

Definition

Let $U, V \in \operatorname{Neigh} \mathbf{A}$.

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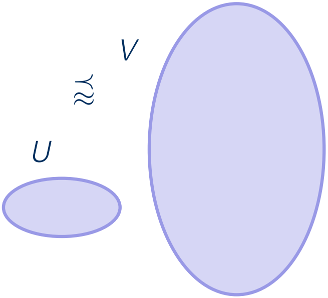

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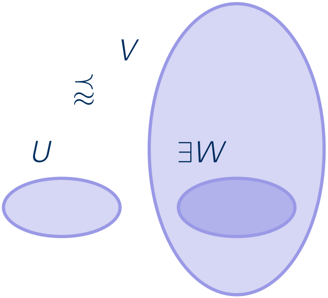

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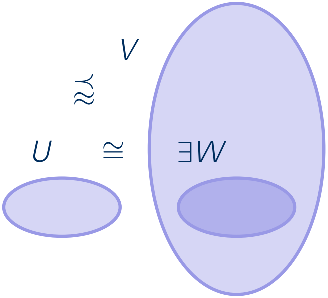

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Lemma

Let $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$, $U \in \operatorname{Neigh} \mathbf{A}$.

- $(\operatorname{Neigh} \mathbf{A}, \lessapprox)$ *is a quasiordered set.*
- A *finite* $\implies \lessapprox \cap \gtrapprox = \cong$.
- $U \lessapprox V \Leftrightarrow \exists f, g \in \operatorname{Clo}^{(1)}(\mathbf{A}) \big(\quad f[A] \subseteq V, g[A] \subseteq U$ *and* $\forall u \in U (\ g(f(u)) = u) \big)$.

Outline

Polynomial expansions?

$$\mathbf{A} = \langle A; F\rangle \quad \rightsquigarrow \quad \mathbf{A}_A := \left\langle A; F \cup \left\{ c_a^{(0)} \;\middle|\; a \in A \right\} \right\rangle$$

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$\mathbf{A}$ finite $\implies$ polynomial operations instead of term op's.

Neighbourhoods of distributive lattices

Definition

$\mathbf{D} = \langle D; \wedge, \vee \rangle$ (distributive) lattice, $a, b \in D$. Set
$e_{a,b}(x) := a \vee (b \wedge x)$ for $x \in D$.

Lemma

For bounded distributive lattices $\mathbf{D}$

1. $$\mathrm{Clo}^{(1)}(\mathbf{D}_D) = \mathrm{Idem}\,\mathbf{D}_D = \{\, e_{a,b} \mid a, b \in D\} = \{\, e_{a,b} \mid a \leq b\} \subseteq \mathrm{Hom}(\mathbf{D}, \mathbf{D})$$
2. $\mathrm{im}\, e_{a,b} = [a, a \vee b]$
3. $\mathrm{Neigh}\,\mathbf{D}_D = \{\, [a, b] \mid a, b \in D, a \leq b\}$

Irreducibility of distributive lattices

Lemma (irreducibility criterion)

A finite algebra $\mathbf{A}$ *is irreducible iff*
$\mathrm{Clo}^{(1)}(\mathbf{A}) \setminus \mathrm{Sym}\, A \in \mathrm{Sub}\left(\mathbf{A}^{A}\right)$.

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1. $\mathrm{Clo}^{(1)}(\mathbf{D}_D) \setminus \mathrm{Sym}\, D = \left\{ e_{a,b} \mid (a,b) \in D^2 \setminus \{(0,1)\}\right\}$
2. $\mathbf{D}$ *finite:* $\mathbf{D}_D$ *irreducible* $\iff$ 0 $\bigwedge$*-irreducible and* 1 $\bigvee$*-irreducible*

Strictly irreducible neighbourhoods

Proposition

For a finite distributive lattice $\mathbf{D}$*, we have*

$$\mathrm{Irr}^*\left(\mathbf{D}_D\right) = \mathrm{Irr}\left(\mathbf{D}_D\right) = \left\{ [a, b] \;\middle|\; a < b, a \bigvee\text{-irr.}, b \bigwedge\text{-irr. in } [a, b] \right\}.$$

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Proof.

Let $a<b$, $U:=\operatorname{im} e_{a,b}=[a,b]$. Then $e_{a,b}\in\operatorname{Hom}\left(\mathbf{D},\mathbf{D}\right) \implies$

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Let $a < b$, $U := \operatorname{im} e_{a,b} = [a,b]$. Then $e_{a,b} \in \operatorname{Hom}(\mathbf{D},\mathbf{D}) \implies \mathbf{D}_D|_U = \mathbf{U}_U|_U$.

$$\begin{aligned} U \text{ irr. in } \mathbf{D}_D \iff \mathbf{D}_D|_U = \mathbf{U}_U|_U \text{ irr.} &\iff \mathbf{U}_U \text{ irr. polyn. exp. of } \mathbf{U} = [a,b]_{\mathbf{D}} \\ &\iff a \bigvee\text{-irr., } b \bigwedge\text{-irr. in } [a,b]_{\mathbf{D}}. \end{aligned}$$

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$$\iff a \bigvee\text{-irr.}, b \bigwedge\text{-irr. in } [a, b]_{\mathbf{D}}.$$

$e_{a,b} \in \operatorname{Hom}(\mathbf{D}, \mathbf{D}) \implies$

$$\restriction_U : \operatorname{Inv}^{(m)} \mathbf{D}_D \longrightarrow \operatorname{Inv}^{(m)} \mathbf{D}_D|_U \qquad \text{complete lattice hom.}$$

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Proof.

Let $a < b$, $U := \operatorname{im} e_{a,b} = [a,b]$. Then $e_{a,b} \in \operatorname{Hom}(\mathbf{D},\mathbf{D}) \implies \mathbf{D}_D|_U = \mathbf{U}_U|_U$.

$$U \text{ irr. in } \mathbf{D}_D \iff \mathbf{D}_D|_U = \mathbf{U}_U|_U \text{ irr.} \iff \mathbf{U}_U \text{ irr. polyn. exp. of } \mathbf{U} = [a,b]_{\mathbf{D}}$$
$$\iff a \bigvee\text{-irr.},\, b \bigwedge\text{-irr. in } [a,b]_{\mathbf{D}}.$$

$e_{a,b} \in \operatorname{Hom}(\mathbf{D},\mathbf{D}) \implies$

$$\restriction_U \colon \operatorname{Inv}^{(m)}\mathbf{D}_D \longrightarrow \operatorname{Inv}^{(m)}\mathbf{D}_D|_U \qquad \text{complete lattice hom.}$$

$\operatorname{Idem}\mathbf{D}_D \subseteq \operatorname{Hom}(\mathbf{D},\mathbf{D}) \implies \operatorname{Irr}^*(\mathbf{D}_D) = \operatorname{Irr}(\mathbf{D}_D)$ □

Isomorphic neighbourhoods

Lemma (Characterisation of isomorphic neighbourhoods)

Let $\mathbf{D}$ *be a bounded distributive lattice,* $a \leq b$*,* $c \leq d$*. Then* $[a, b] \cong [c, d]$ *iff one of the following cases is true*

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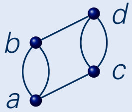

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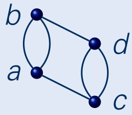

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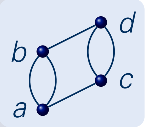

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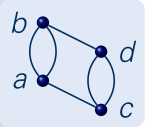

3. $a \parallel c$ *and* $b \parallel d$ *and*

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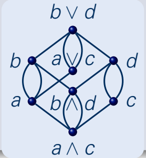

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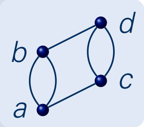

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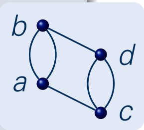

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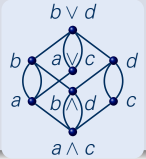

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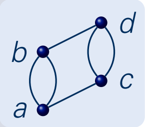

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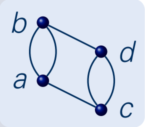

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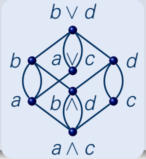

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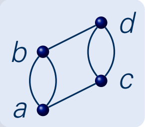

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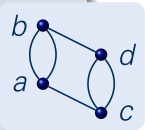

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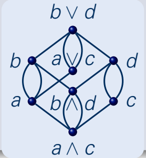

Toy example