Dualizability of Automatic Algebras

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Introduction

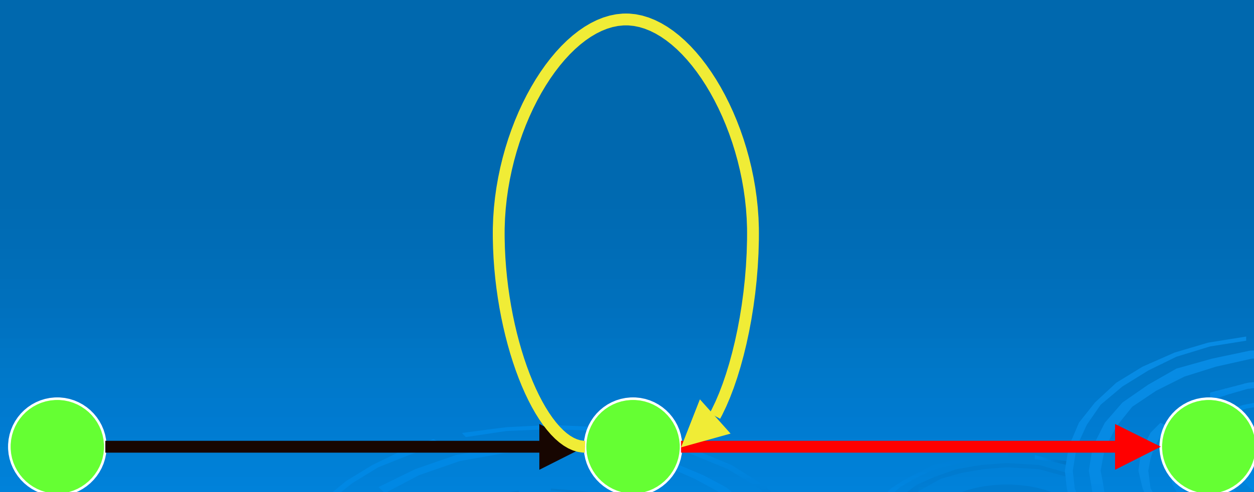
"A dozen easy problems"

- Presentation by McNulty at the 2010 AMS Special Session in St. Paul
- Contains 6 new and 6 revived problems in Universal Algebra
- Important footnote:

"Well, easily formulated …"

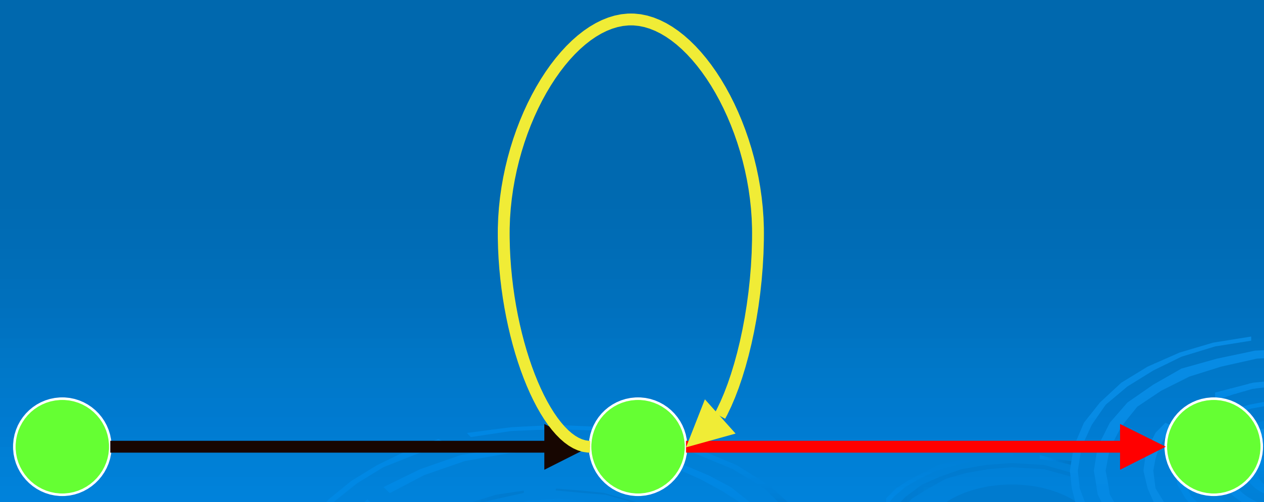
Question 6
Automatic Algebra Problems

- “Which finite automatic algebras are dualizable?”
- “Is the automatic algebra drawn below dualizable?”



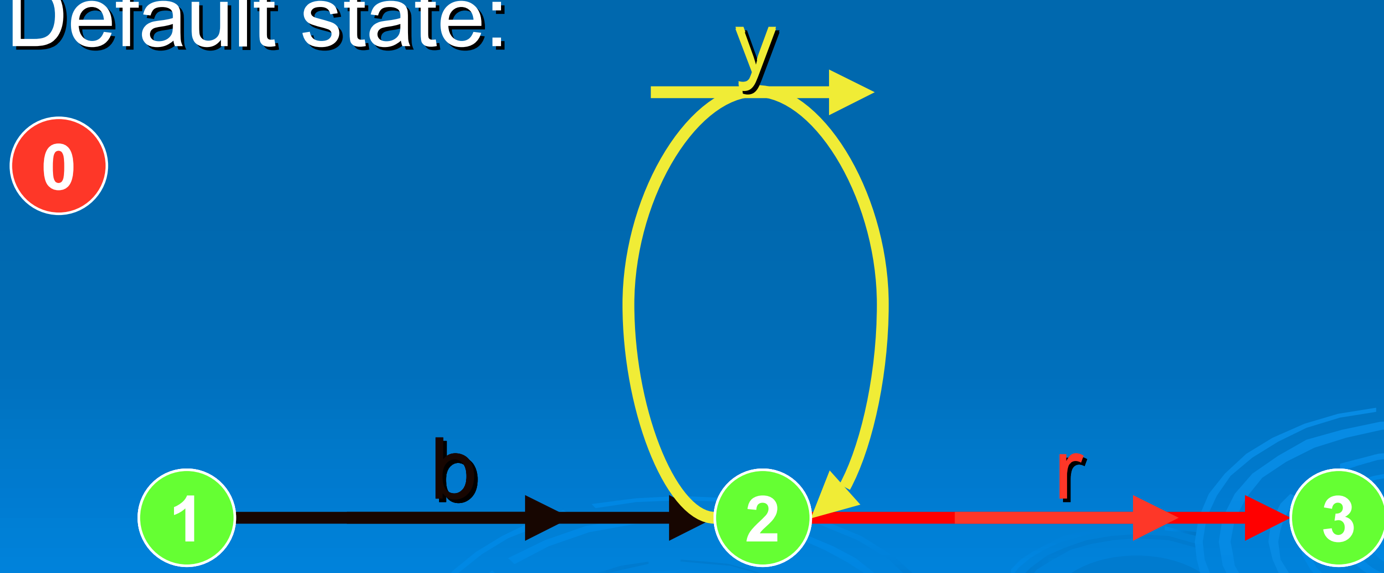
What are Automatic Algebras?

- Start with a (partial) automaton without initial and terminal states
- Transition labels are colour coded (for this presentation)



Automatic Algebras - Elements

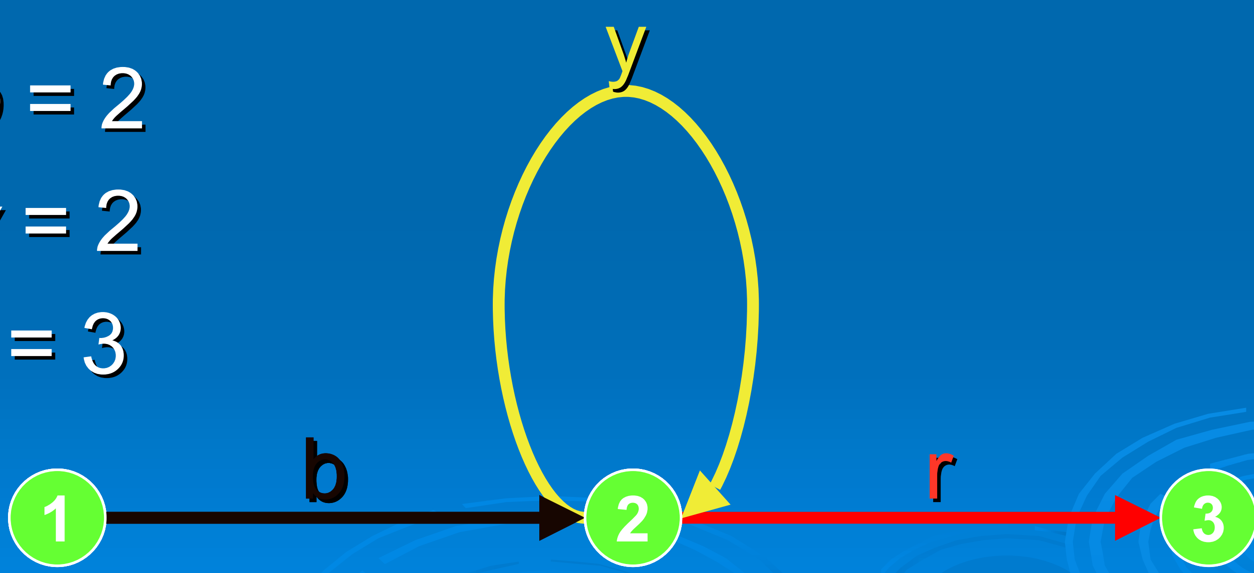
- States:
- Transition types (letters):
- Default state:



Automatic Algebras
One binary operation $\circ$

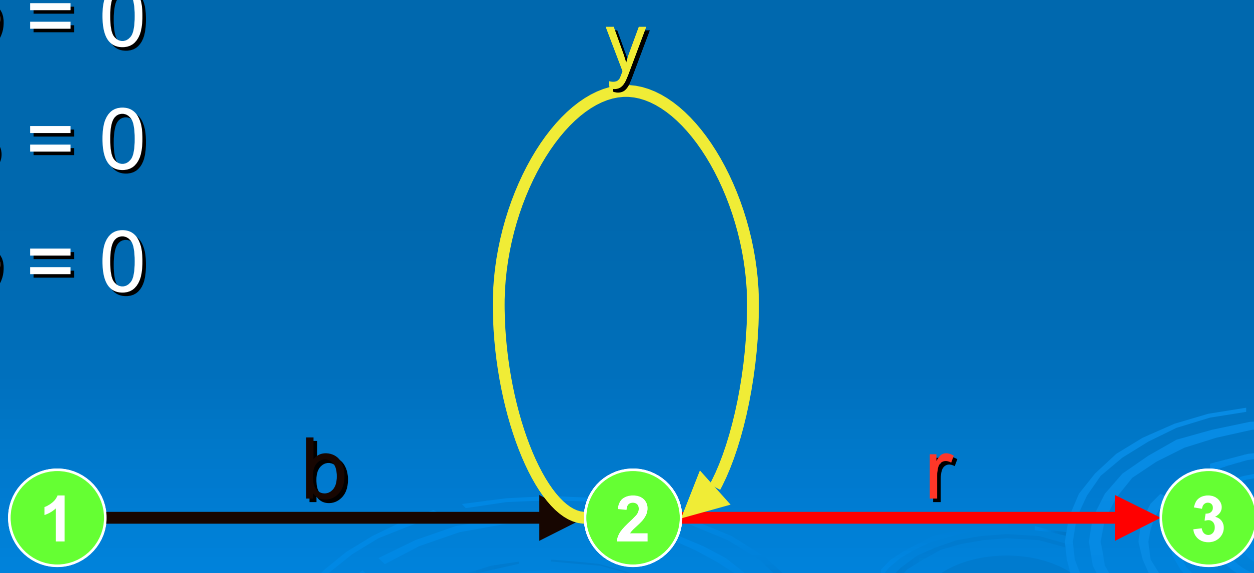
- If letter *a* induces a transition from state *n* to state *m*, let $n \circ a = m$

- $1 \circ b = 2$
- $2 \circ y = 2$
- $2 \circ r = 3$



Automatic Algebras
One binary operation ∘

- All other products evaluate to the default state 0
- $0 \circ a = 0$
- $b \circ b = 0$
- $2 \circ 3 = 0$
- $2 \circ b = 0$



What are Dualities?

- Well ….
- Ross Willard once introduced them like this …

R. Willard, "Four unsolved problems in congruence-permutable varieties", Nashville, 2007

Definition

A finite algebra ~~A~~ $\underline{\mathbf{M}}$ is **dualizable** if

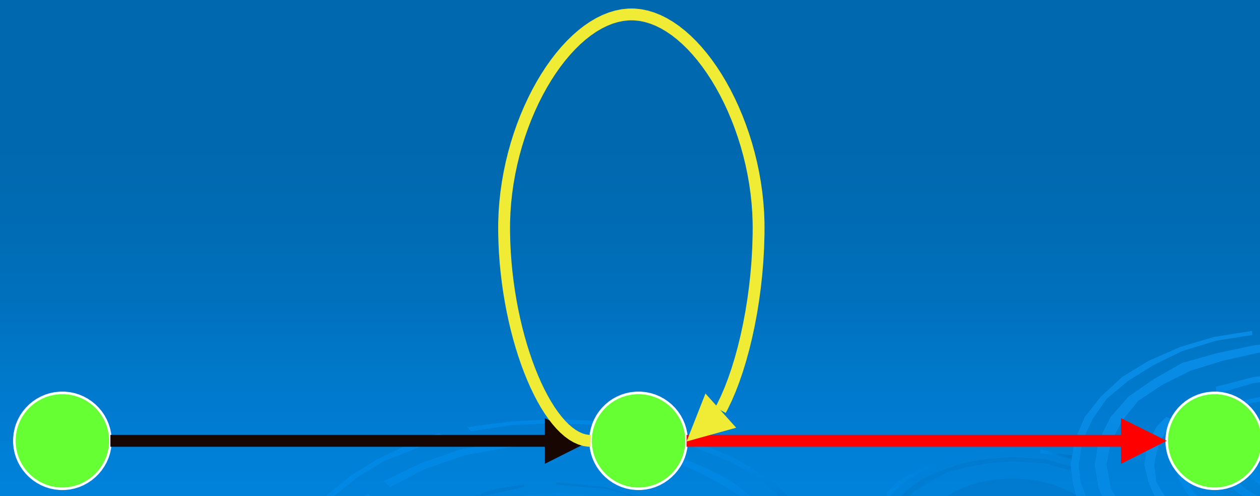
- there exists an "alter ego" $\underset{\sim}{\mathbf{M}}$ ...
- ...partial operations ...relations ...discrete topology ...
- ...**ISP** and $\mathbf{IS_cP}^+$ ...
- ...contravariant hom-functors ...
- ...dual adjunction (D, E, e, ε) ...
- **AARRRGGHH!!! STOP THE INSANITY!!**

Why dualities for automatic algebras?

- Lyndon’s example of an algebra that is not finitely based is an automatic algebra
- Automatic algebras have been examined for finite basedness by Boozer and have given interesting examples
- Dualizability is another finiteness condition
- Its relationship with finite basedness is not well-known

Why ? (2)

- This algebra is non-finitely based, but “just fails” to be inherently non-finitely based

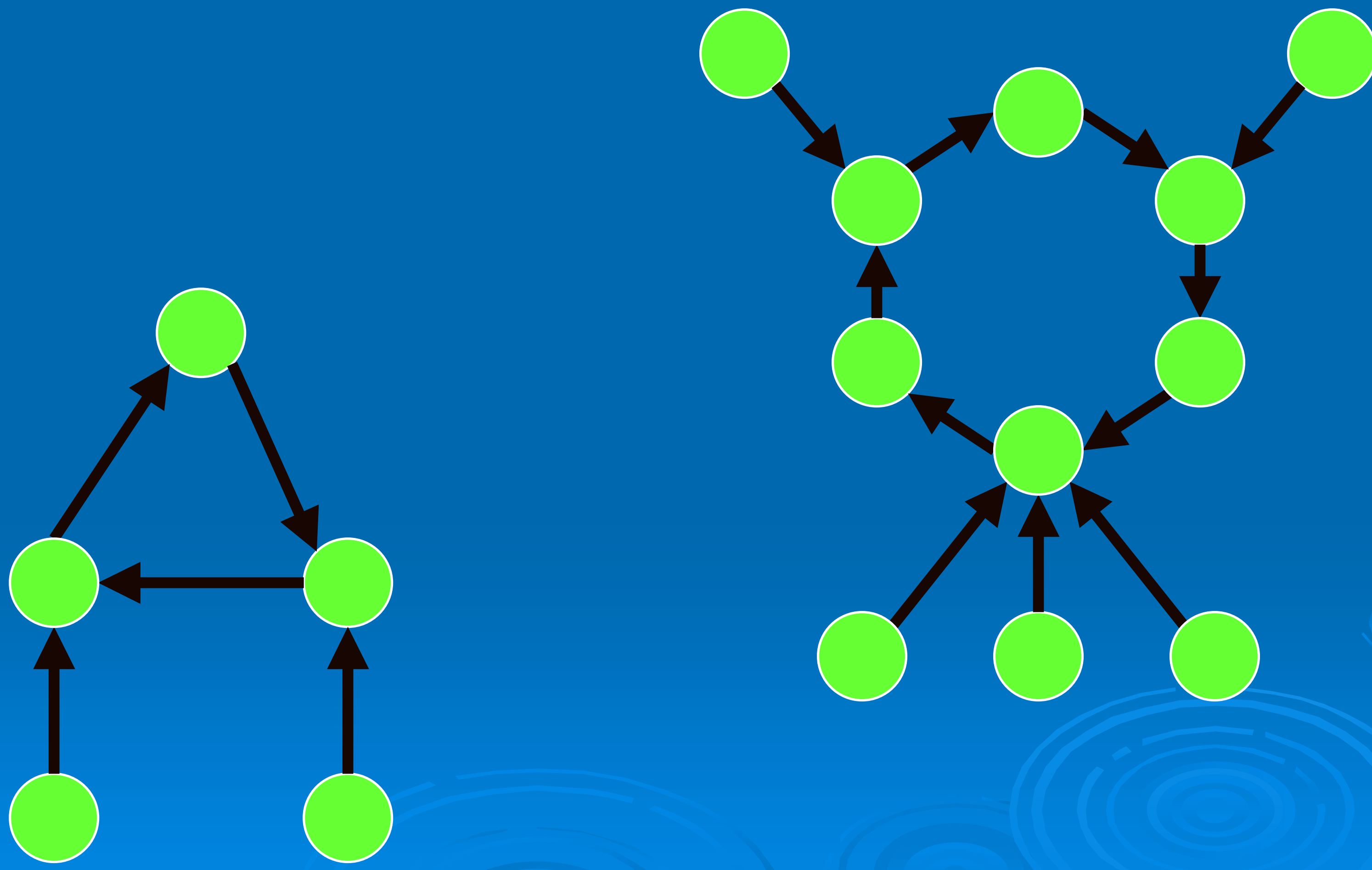


Results

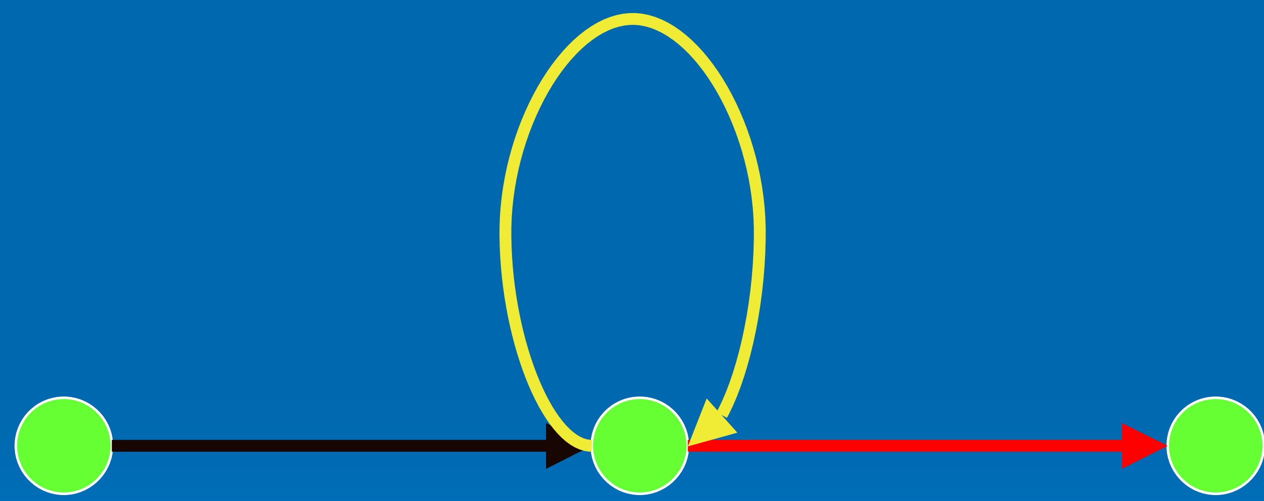
- If an automatic algebra is dualizable then every letter acts as whiskery cycles

- If a letter acts otherwise, it is *inherently non-dualizable* (i.e. all its superalgebras are non-dualizable)

Whiskery Cycles



Inherently non-dualizable

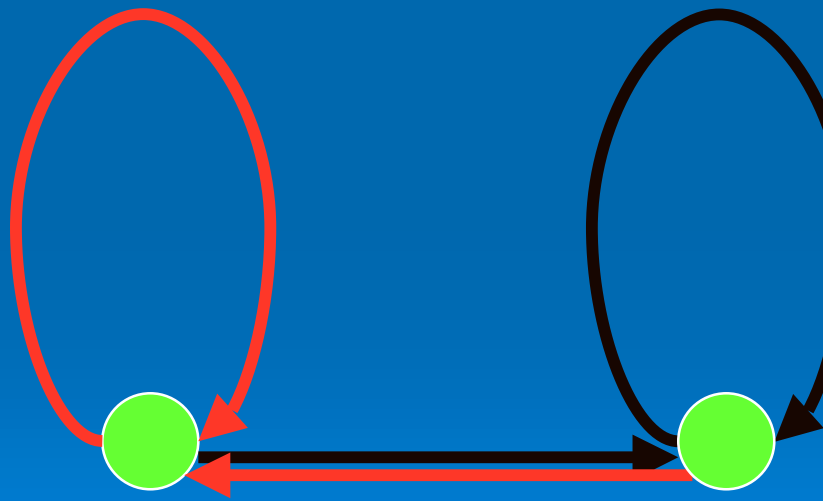


Classification

- An algebra with exactly one letter is dualizable if and only if the letter acts as whiskery cycles

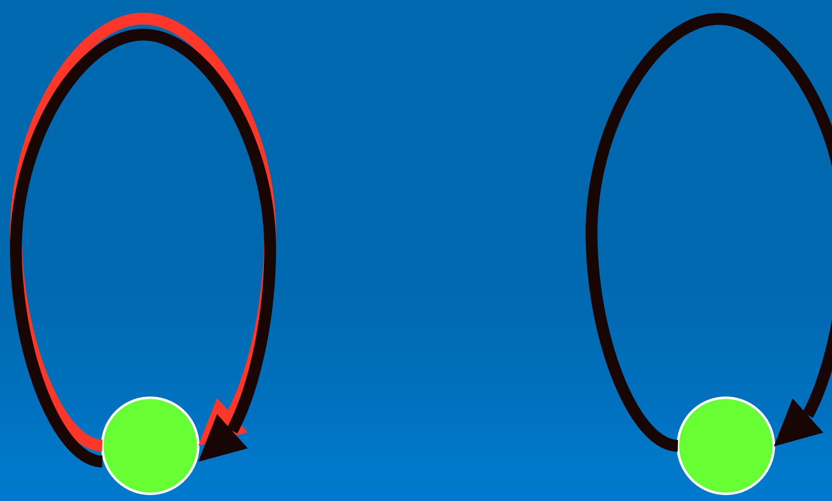
Results

- If each letter acts as a total constant function, then the algebra is dualizable



Results

- If each edge is a loop, then the algebra is dualizable

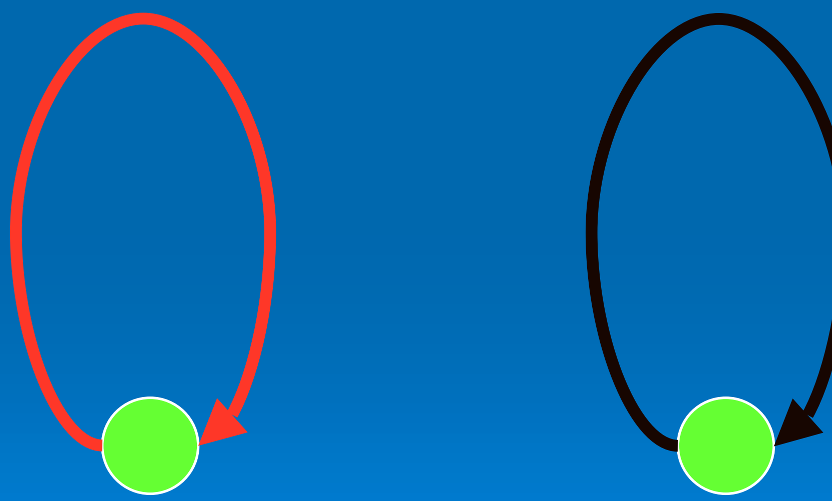


Classification

- All automatic algebras with exactly one non-zero state are dualizable
- An algebra with exactly two non-zero states is dualizable if and only if
  - each letter acts as whiskery cycles
  - the algebra is pseudo-commutative, i.e. satisfies $((x \circ y) \circ z) \circ w = ((x \circ z) \circ y) \circ w$

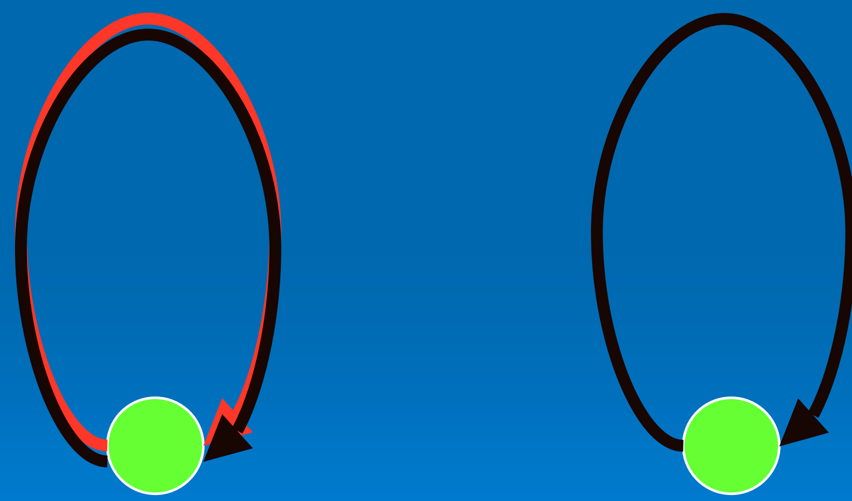
Classification

- All dualizable automatic algebras with 2 states and at least 2 letters are (essentially)



Classification

- All dualizable automatic algebras with 2 states and at least 2 letters are (essentially)



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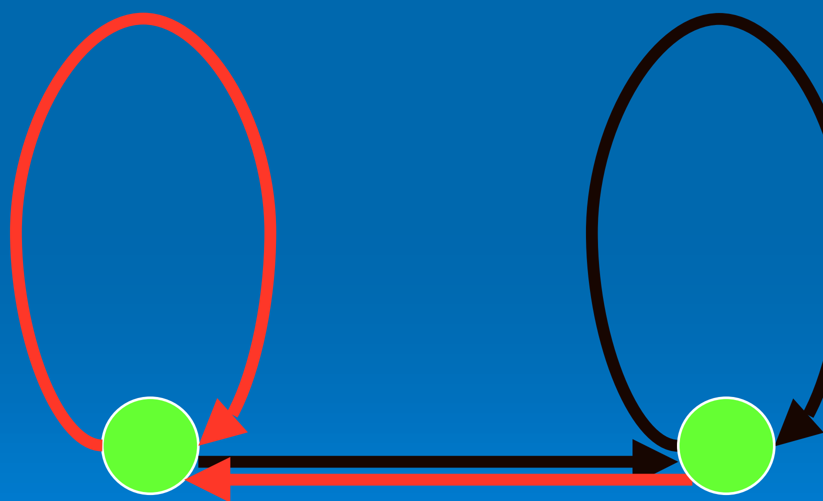

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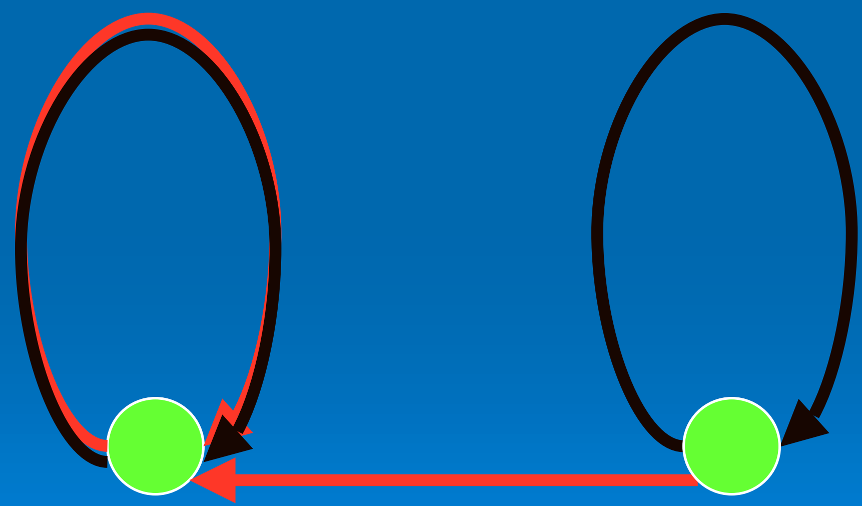

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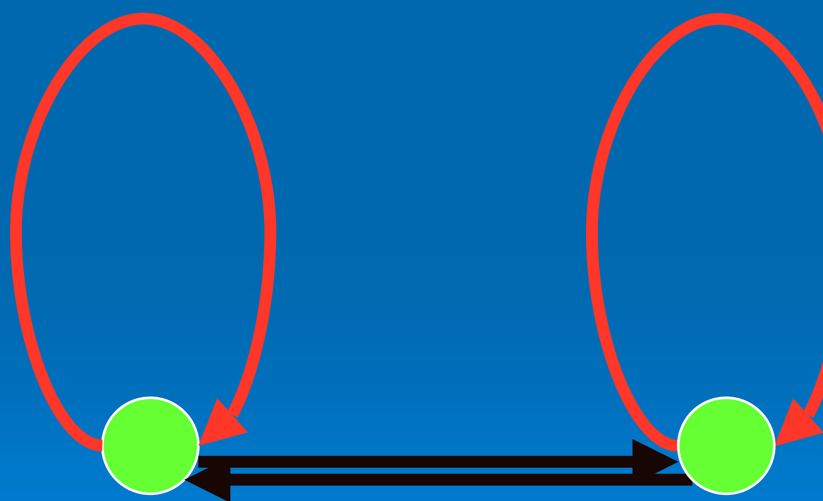
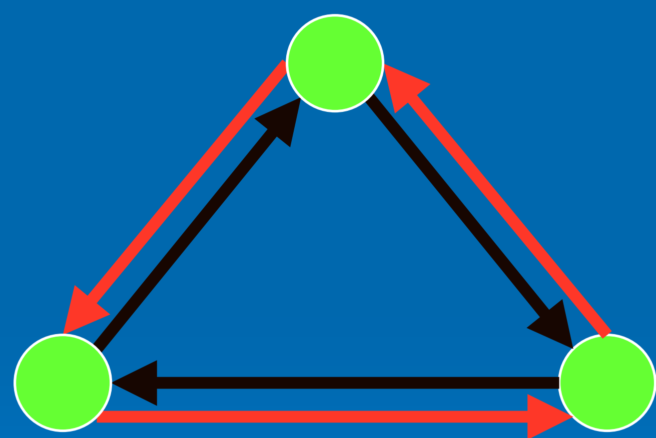


Unfortunately …

- whiskery cycles + pseudo commutativity does not work if there are more states



Non-dualizable (but not inherently so!)

Commutative permutations

- Each letter acts as a permutation of the non-zero states and all such actions commute

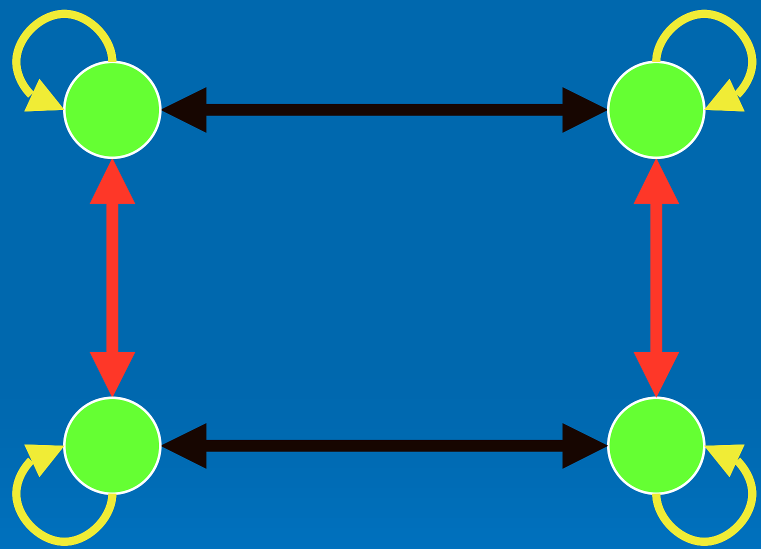
- So we can identify the letters with members of an Abelian group of permutations

Result

- For an automaton as above, if the letter actions form a coset of the corresponding Abelian permutation group, then the algebra is dualizable
- In particular,
  - if there is just one permutation, the algebra is dualizable
  - if the letter actions form an Abelian group the algebra is dualizable

Unfortunately …

- the converse is false



dualizable

Negative Results

- For connected automata, “certain” non-singleton cosets must be present among the letter actions in dualizable algebras

- In particular, if the algebra has at least two distinctly acting letters, but less then needed to form the smallest non-trivial coset, the algebra is not dualizable

How difficult is the (easily formulated) problem ?

- We are able to construct a chain of superalgebras that are alternatively dualizable and non-dualizable

- Such a chain has previously only been constructed in the class of unary algebras

Thank you!