Topological Birkhoff

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Joint work with Michael Pinsker

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Overview

1. Birkhoff's Theorem

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2. Topological Birkhoff
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map $t^{\mathbf{A}}$ to $t^{\mathbf{B}}$, for all τ-terms t.
If well-defined, call this map the natural homomorphism from Clo($\mathbf{A}$) $\to$ Clo($\mathbf{B}$).

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1. The natural homomorphism from Clo$(\mathbf{A})$ to Clo$(\mathbf{B})$ exists.
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If $\mathbf{A}$ is infinite, have to replace $\mathsf{HSP}^{\text{fin}}(\mathbf{A})$ by $\mathsf{HSP}(\mathbf{A})$

and pseudo-varieties by varieties – even when $\mathbf{B}$ is finite.

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Fact

A polymorphism clone of a countable structure Γ is oligomorphic if and only if Γ is ω-categorical, i.e., every countable model of the first-order theory of Γ is isomorphic to Γ.

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Let $\mathbf{A}$, $\mathbf{B}$ be oligomorphic or finite algebras with the same signature. Tfae:

1. The natural homomorphism from Clo$(\mathbf{A})$ to Clo$(\mathbf{B})$ exists and is continuous.
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- It suffices that $\mathbf{B}$ is finitely generated (oligomorphic algebras are finitely generated)

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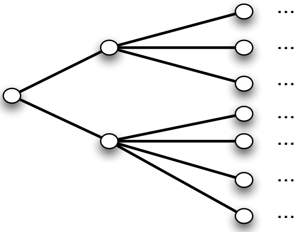

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Consequence: when $\mathbf{A}$ is locally oligomorphic, and G consists of the unary invertible operations in $\overline{\text{Clo}(\mathbf{A})}$, then $\overline{\text{Clo}(\mathbf{A})}^{(k)}/G$ is compact.

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Want to prove: $\mathbf{B} \in \mathsf{HSP}^{\mathsf{fin}}(\mathbf{A})$ if and only if
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Lemma.

For all finite $F \subseteq B$ and all $k \geq 1$ there exists an $m \geq 1$ and $C \in A^{m \times k}$ s.t.
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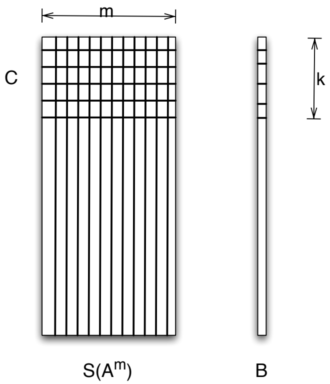

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There is an oligomorphic $\mathbf{A}$ and finite $\mathbf{B}$ with common signature such that $\mathbf{B} \in \mathsf{HSP}(\mathbf{A})$, but $\mathbf{B} \notin \mathsf{HSP}^{\mathsf{fin}}(\mathbf{A})$.

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(Thanks to Keith Kearnes)

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Examples. All homogeneous structures with finite relational signature (e.g. from the talks of Manfred Droste and John Truss!) are ω-categorical.

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Question (B.-Junker): can this be further generalized to topological clones and primitive positive bi-interpretability?

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- a τ-formula $\delta_I(x_1, \dots, x_d)$,
- for each atomic σ-formula $\phi(y_1, \dots, y_k)$ a τ-formula $\phi_I(\overline{x}_1, \dots, \overline{x}_k)$,
- a surjective map $h\colon \delta_I(\Delta^d) \to \Gamma$,

such that for all atomic σ-formulas ϕ and all $\overline{a}_i \in \delta_I(\Delta^d)$

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Definition.

An interpretation is primitive positive (pp) iff all the involved formulas are primitive positive, i.e., of the form

$$\exists x_1,\dots,x_n\,(\psi_1 \wedge \dots \wedge \psi_l)$$

where ψ_i are atomic, i.e. of the form $x = y$ or $R(x_{i_1},\dots,x_{i_k})$ for $R \in \tau$.

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- Δ has a primitive positive interpretation in Γ.

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- Δ has a primitive positive interpretation in Γ.
- For every polymorphism algebra $\mathbf{A}$ of Γ there is an algebra $\mathbf{B} \in \text{HSP}^{\text{fin}}(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}) \subseteq \text{Pol}(\Delta)$.

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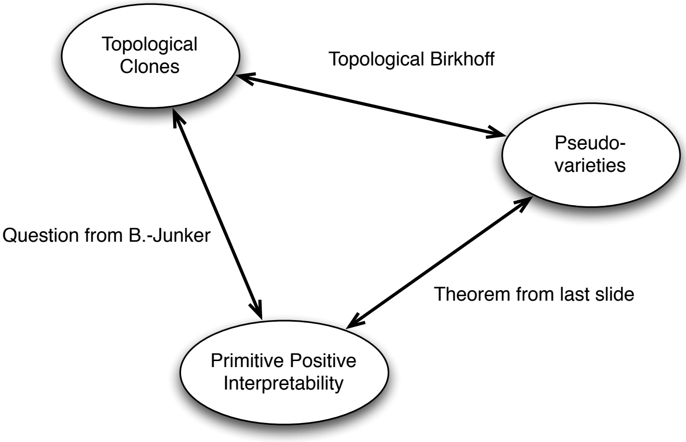

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A reduct of a structure Δ is a structure obtained from Δ by dropping some of the relations from Δ.

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Theorem.

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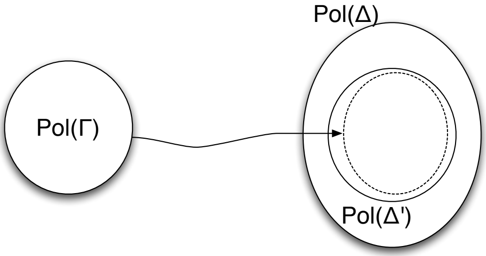
- Δ is the reduct of a finite or ω-categorical structure Δ' such that there exists a continuous homomorphism from Pol(Γ) to Pol(Γ') whose image is dense in Pol(Δ').



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Say that mutually interpretable Γ and Δ are pp bi-interpretable iff the coordinate maps h_1 and h_2 of the pp interpretations are such that

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- Γ and Δ are primitive positive bi-interpretable;
- Γ has a polymorphism algebra $\mathbf{A}$ and Δ has a polymorphism algebra $\mathbf{B}$ such that $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A}) = \mathrm{HSP}^{\mathrm{fin}}(\mathbf{B})$.

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  $\xi \colon \operatorname{Aut}(\Gamma) \to \operatorname{Aut}(\Delta)$ defined by $f \mapsto f|_P$ is continuous homomorphism whose image is dense in $\operatorname{Aut}(\Delta)$.
  But ξ is not surjective! (D. Macpherson).

Constraint Satisfaction Problems

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Theorem 2.
For ω-categorical Γ, the complexity of CSP(Γ) only depends on the topological polymorphism clone of Γ.

(answering question from Fields-Institute Summer on CSPs and Algebra'11)

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Since d is clearly unique for each f, setting $\xi(f) := \pi^k_d$ defines a function ξ from Pol(Γ) onto $\mathbf{1}$. Straightforward: ξ is continuous homomorphism.

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- and the Henson graphs (Herwig'98).

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But this doesn't answer my questions for polymorphism clones:

- when does the abstract clone determine the topological one?
- does the complexity of CSP(Γ) only depend on the abstract clone of Γ?

Reference

Topological Birkhoff, Manuel Bodirsky and Michael Pinsker,
http://arxiv.org/abs/1203.1876, 2012.