CSP dichotomy for special oriented trees

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Outline

$\mathbb{H}$-colouring problem

Let $\mathbb{H}$ be a directed graph.

Definition

CSP($\mathbb{H}$), or the $\mathbb{H}$-colouring problem, is the following decision problem:

INPUT: a digraph $\mathbb{G}$

QUESTION: Is there a homomorphism $\mathbb{G} \to \mathbb{H}$?

Conjecture (Feder, Vardi'99)

For every $\mathbb{H}$, CSP($\mathbb{H}$) is in P or NP-complete.

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Polymorphisms

Let $\mathbb{H} = (H, \to)$ be a digraph.

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An operation $f : H^n \to H$ is a polymorphism of $\mathbb{H}$ if whenever $\forall i : a_i \to b_i$, then $f(a_1, \dots, a_n) \to f(b_1, \dots, b_n)$.

$$\begin{array}{cccccc} f(a_1 & a_2 & \dots & a_n) & = & a \\ \downarrow & \downarrow & & \downarrow & \Longrightarrow & \downarrow \\ f(b_1 & b_2 & \dots & b_n) & = & b \end{array}$$

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The *algebra of (idempotent) polymorphisms* of $\mathbb{H}$:

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Algebraic dichotomy

Let $\mathbb{H}$ be a core digraph.

Theorem (Jeavons, Bulatov, Krokhin'00-05)

If **alg**$\mathbb{H}$ *is not Taylor, then CSP($\mathbb{H}$) is NP-complete.*

Taylor algebra = $\mathcal{V}(\mathbf{A})$ satisfies some nontrivial maltsev condition

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An important tractable case:

Theorem ("Bounded Width Theorem", Barto, Kozik'08)

If **alg**$\mathbb{H}$ *is SD($\wedge$), then* $\mathbb{H}$ *has bounded width ($\Rightarrow$CSP($\mathbb{H}$) is in P).*

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Levels, minimal paths

Let $\mathbb{H}$ be an oriented tree.

- we can assign levels to its vertices
- maximum level = *height* of $\mathbb{H}$.

An oriented path $\mathbb{P}$ is minimal, if its initial vertex has level 0, terminal vertex level k, and for all other vertices $0 < \mathrm{level}(v) < k$

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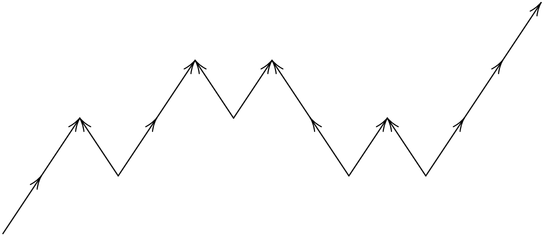

Special trees

Definition

Let $\mathbb{T}$ be an oriented tree of height 1. A $\mathbb{T}$-special tree is an oriented tree obtained from $\mathbb{T}$ by replacing all edges by minimal paths of the same height (preserving orientation).

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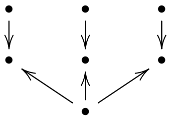
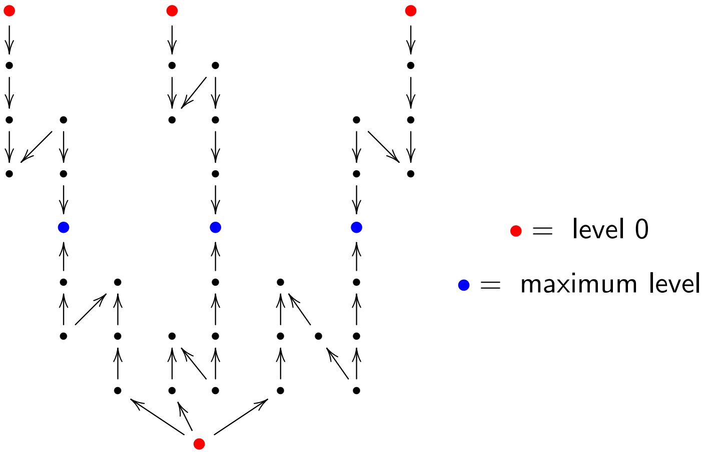
$\mathbb{T} =$ 

Example of a special triad



Problem (Barto, Kozik, Maróti, Niven)

Is this the smallest NP*-complete oriented tree?*

History of special trees

- (Hell, Nešetřil, Zhu'90): a very specific subclass of triads, the special triads; constructing a small NP-complete oriented tree
- (Barto, Kozik, Maróti, Niven'08): dichotomy for special triads; tractable cases are easy – either majority polymorphism or width 1
- (Barto, JB'10): dichotomy for special polyads; tractable ones have BW (Taylor $\Rightarrow SD(\wedge)$), but are not so easy
  + we can generate nice (counter-)examples in trees
- (JB'12): dichotomy for a larger class of special trees; a new proof using absorption techniques

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New result

Proposition (JB'12)

Let $\mathbb{T}$ be an oriented tree of height 1 satisfying one of these conditions:

1. *maximum degree of $\mathbb{T}$ is ≤ 3*
2. *$\mathbb{T}$ has at most 3 vertices of degree > 2.*

Then the CSP dichotomy holds for $\mathbb{T}$-special trees.

More specifically, for all $\mathbb{T}$-special trees $\mathbb{H}$, if **alg**$\mathbb{H}$ *is Taylor, then it is SD($\wedge$).*

Strategy of proof:

- Absroption Theorem $\Rightarrow$ **alg**$\mathbb{H}$ can't have many absorption-free subalgebras...
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Absorption

Definition

A subalgebra $\mathbf{C} \leq \mathbf{A}$ is absorbing ($\mathbf{C} \trianglelefteq \mathbf{A}$), if there exists an idempotent t such that

$$\begin{aligned} t(C, C, \dots, C, A) &\subseteq C, \\ t(C, C, \dots, A, C) &\subseteq C, \\ &\vdots \\ t(A, C, \dots, C, C) &\subseteq C. \end{aligned}$$

Example: A (finite) algebra $\mathbf{A}$ has a near-unanimity term iff $\{a\} \trianglelefteq \mathbf{A}$ for every $a \in A$.

$\mathbf{A}$ is absorption-free if it has no proper absorbing subalgebras.

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Some facts about absorption

Theorem ("Absorption Theorem", Barto, Kozik'10)

$\mathbf{A}, \mathbf{B}$ *finite algebras in a Taylor variety,* $E \leq_S \mathbf{A} \times \mathbf{B}$ *linked. Then there exist* $\mathbf{C} \trianglelefteq \mathbf{A}$, $\mathbf{D} \trianglelefteq \mathbf{B}$ *such that* $E \restriction C \times D = C \times D$.

linked = connected as a bipartite graph

Lemma (Barto, Kozik)

Let $\mathbf{A}$ *be a finite idempotent algebra. Then* $\mathbf{A}$ *is SD*$(\wedge)$ *iff all absorption-free subalgebras of* $\mathbf{A}$ *are SD*$(\wedge)$.

Proof: Follows from the Bounded Width algorithm.

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Sketch of the proof

Let $\mathbb{T} = (T, E)$ be an oriented tree of height 1,
$\mathbb{H}$ a $\mathbb{T}$-special tree such that $\mathbf{alg}\mathbb{H}$ is Taylor.

- $\mathbf{A} = \{\text{vertices of level } 0\} \leq \mathbf{alg}\mathbb{H}$
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- $\mathbf{alg}\mathbb{H}$ is $SD(\wedge)$ iff both $\mathbf{A}$ and $\mathbf{B}$ are $SD(\wedge)$ (this is what makes the trees "special")
- $E \leq_S \mathbf{A} \times \mathbf{B}$ (E is pp-definable), E is a tree

Lemma

Let $\mathbf{A}$, $\mathbf{B}$ be finite idempotent algebras in a Taylor variety and $E \leq_S \mathbf{A} \times \mathbf{B}$ a tree such that

$$E^+(a) \text{ and } E^-(b) \text{ are } SD(\wedge)\ \forall a \in A\, \forall b \in B.$$

Then $\mathbf{A}$ and $\mathbf{B}$ are $SD(\wedge)$.

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It remains to prove that E-neigbourhoods of singletons are $SD(\wedge)$. For that we have an ad hoc construction:

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Let $\mathbf{D} \leq E^+(a)$ be absorption-free. There exists a binary idempotent polymorphism $\star$ of $\mathbb{H}$ such that $\star \restriction D$ is commutative (i.e., a 2-wnu).

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Open problems

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Prove that Taylor implies $SD(\wedge)$ for all special trees.

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Can these techniques be adapted for general orientes trees? Maybe just for triads?

Problem

Was that the smallest $\mathtt{NP}$-complete oriented tree?

Thanks

Thank you for your attention!