Lattice polynomial functions and their use in qualitative decision making

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Decision making **DM**

Main Problem: Model preference!

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Model: R on $X_1 \times \cdots \times X_n$ is represented by $f\colon X_1 \times \cdots \times X_n \to X$:

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$$\mathbf{x} R \mathbf{y} \iff f(\mathbf{x}) \leq f(\mathbf{y})$$

Limitation: The role of local preferences is not explicit!

Aggregation: $x_1, \ldots, x_n \quad \longrightarrow \quad y = A(x_1, \ldots, x_n)$

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Let X be a scale (bounded chain).

An aggregation function on X is a mapping $A\colon X^n \to X$ such that:

1. A is order-preserving: for every $\mathbf{x}, \mathbf{y} \in X^n$

$$\mathbf{x} \leq \mathbf{y} \implies A(\mathbf{x}) \leq A(\mathbf{y})$$

2. A preserves the boundaries:

$$\inf_{\mathbf{x} \in X^n} A(\mathbf{x}) = \inf X \quad \text{and} \quad \sup_{\mathbf{x} \in X^n} A(\mathbf{x}) = \sup X.$$

Traditionally: X is a real interval $\mathbb{I} \subseteq \mathbb{R}$, e.g., $\mathbb{I} = [0, 1]$.

Aggregation in decision making **DM**

Numerical representation of relations: $f\colon X_1 \times \cdots \times X_n \to \mathbb{I} \subseteq \mathbb{R}$:

$$\mathbf{x}R\mathbf{y} \quad \Longleftrightarrow \quad f(\mathbf{x}) \leq f(\mathbf{y})$$

DM: Preference on criteria i is represented by a local utility function

$$\varphi_i\colon X_i \to \mathbb{I}.$$

Preference on $X_1 \times \cdots \times X_n$ is represented by an overall utility function:

$$F(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

where $A\colon \mathbb{I}^n \to \mathbb{I}$ is an aggregation function.

Examples of aggregation functions:

1. **Arithmetic means:** For $\mathbf{x} \in \mathbb{I}^n$,

$$AM(\mathbf{x}) := \frac{1}{n} \sum_{1 \leq i \leq n} x_i$$

2. **Weighted arithmetic means:** For $\mathbf{x} \in \mathbb{I}^n$ and $\sum w_i = 1$,

$$WAM(\mathbf{x}) := \sum_{1 \leq i \leq n} w_i x_i$$

3. **Choquet integrals:** For $\mathbf{x} \in \mathbb{I}^n$,

$$C(\mathbf{x}) := \sum_{I \subseteq \{1,\ldots,n\}} a_I \cdot \bigwedge_{i \in I} x_i$$

Qualitative decision making **QDM**

In the qualitative approach:

The underlying sets $X_1, \ldots, X_n$ and X are finite chains (ordinal scales),

$$\text{e.g.,} \quad X = \{\text{very bad, bad, satisfactory, good, very good}\}$$

QDM: Preference relation on X_i is represented by

$$\varphi_i \colon X_i \to X.$$

Preference relation on $X_1 \times \cdots \times X_n$ is represented by

$$F(x_1, \ldots, x_n) := A(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

where $A \colon X^n \to X$ is an aggregation function.

Capacities

Let X be a chain with least and greatest elements 0 and 1, respectively.

A capacity is a mapping $v\colon 2^{[n]} \to X$, $[n] = \{1, \ldots, n\}$, such that

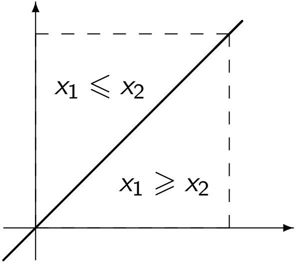
1. $v(I) \leq v(J)$ whenever $I \subseteq J$,
2. $v(\varnothing) = 0$ and $v([n]) = 1$.

Order simplexes of X^n

Let σ be a permutation on $[n] = \{1, \ldots, n\}$ $\quad (\sigma \in S_n)$

$$X^n_\sigma = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in X^n \ : \ x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)} \right\}$$

Example: *$X = [0, 1]$ and $n = 2$*



$2! = 2$ permutations (2 simplexes)

Sugeno integral

The (discrete) Sugeno integral on X w.r.t. v is defined by

$$\mathcal{S}_v(\mathbf{x}) := \bigvee_{i\in[n]} v(\{\sigma(i),\dots,\sigma(n)\}) \wedge x_{\sigma(i)}$$

for every $\mathbf{x} \in X^n_\sigma = \big\{(x_1,\dots,x_n) \in X^n \ : \ x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\big\}$.

Example

If $x_3 \leqslant x_1 \leqslant x_2$, then $x_{\sigma(1)} = x_3$, $x_{\sigma(2)} = x_1$, $x_{\sigma(3)} = x_2$, and

$$\mathcal{S}_v(x_1,x_2,x_3) = (\underbrace{v(\{1,2,3\})}_{=1} \wedge x_3) \vee (v(\{1,2\}) \wedge x_1) \vee (v(\{2\}) \wedge x_2)$$

Qualitative decision making **QDM**

Setting:

1. n criteria on finite chains $X_1, \dots, X_n$
2. scores in a common finite chain X by local utility functions

$$\varphi_i \colon X_i \to X$$

   We will assume that each φ_i is **order-preserving**.

3. Preference relation on $X_1 \times \cdots \times X_n$ is represented by

$$F(x_1, \dots, x_n) := A(\varphi_1(x_1), \dots, \varphi_n(x_n))$$

   where $A \colon X^n \to X$ is a Sugeno integral. We shall refer to these overall utility functions as Sugeno utility functions.

Outline

1. Preliminaries: Sugeno integrals as lattice polynomial functions.
2. Characterizations of lattice polynomial functions.

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3. Generalization of polynomial functions: Sugeno utility functions.
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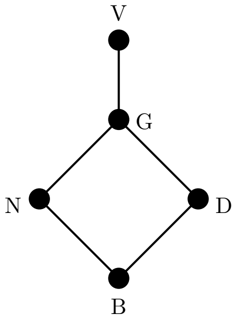
2. Characterizations of lattice polynomial functions.
3. Generalization of polynomial functions: Sugeno utility functions.
4. Sugeno utility functions: characterizations and factorizations.
5. Axiomatic approach to qualitative decision-making **QDM**.
6. Further research directions and open problems.

Preliminaries

Let X be a distributive (finite) lattice with

1. operations $\wedge$ and $\vee$,
2. least and greatest elements 0 and 1, respectively.



Lattice polynomial functions

A (lattice) polynomial function (on X) is any map $p : X^n \to X$, $n \geq 1$, obtainable by finitely many applications of the rules:

1. The projections $\mathbf{x} \mapsto x_i$, $i \in [n]$, and the constant functions $\mathbf{x} \mapsto c$, $c \in X$, are polynomial functions.
2. If $f : X^n \to X$ and $g : X^n \to X$ are polynomial functions, then $f \wedge g$ and $f \vee g$ are polynomial functions.

Example

$$\mathrm{median}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1)$$

Representations: Disjunctive Normal Form

A function $f\colon X^n \to X$ has a disjunctive normal form (**DNF**) if

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(a_I \wedge \bigwedge_{i \in I} x_i \Big).$$

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Proposition (Goodstein'67)

A function $p\colon X^n \to X$ is a polynomial function **iff** it has the **DNF**:

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(p(\mathbf{1}_I) \wedge \bigwedge_{i \in I} x_i \Big)$$

where $\mathbf{1}_I$ denotes the "characteristic tuple" of $I \subseteq [n]$.

Sugeno integrals as lattice polynomial functions

The Sugeno integral on a chain X w.r.t. $v\colon 2^{[n]} \to X$ is defined by

$$\mathcal{S}_v(\mathbf{x}) := \bigvee_{i \in [n]} v(\{\sigma(i), \ldots, \sigma(n)\}) \wedge x_{\sigma(i)}$$

for every $\mathbf{x} \in X^n_\sigma = \big\{(x_1, \ldots, x_n) \in X^n \ : \ x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\big\}$.

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Theorem (Marichal)

A function $q : X^n \to X$ is the Sugeno integral $\mathcal{S}_v$ **iff**

$$q(\mathbf{x}) = \bigvee_{I\subseteq[n]} \big(v(I) \wedge \bigwedge_{i\in I} x_i\big).$$

Since, $q(\mathbf{1}_I) = v(I)$, and $v(\varnothing) = 0$ and $v([n]) = 1$, Sugeno integrals coincide with **idempotent** polynomial functions: $q(x,\ldots,x) = x$.

General properties of polynomial functions

Fact

Every polynomial function (in part., Sugeno integral) is **order-preserving**.

However...

The function $f(0) = f(a) = 0$ and $f(1) = 1$ is order-preserving on $\{0, a, 1\}$, **but** it is not a polynomial function, hence not a Sugeno integral!

Median decomposability (Marichal)

For $c \in X$ and $i \in [n]$, set $\mathbf{x}_i^c = (x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n)$.
A function $f \colon X^n \to X$ is median decomposable if for each $i \in [n]$

$$f(\mathbf{x}) = \mathrm{median}\big(\, f(\mathbf{x}_i^0)\,,\ x_i\,,\ f(\mathbf{x}_i^1)\,\big), \qquad \text{for every } \mathbf{x} \in X^n.$$

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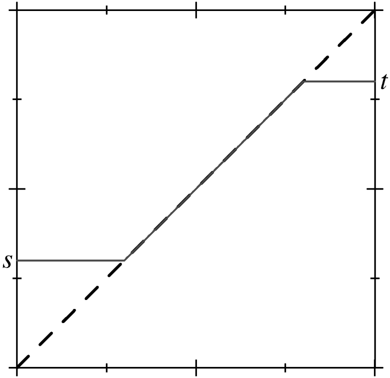

$t = f(\mathbf{x}_i^1)$

$s = f(\mathbf{x}_i^0)$

Characterization of polynomial functions

Fact

Every median decomposable function is order-preserving.

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Theorem (Marichal)

A function $p\colon X^n \longrightarrow X$ is

1. a polynomial function **iff** it is median decomposable.
2. a Sugeno integral **iff** it is idempotent and median decomposable.

General characterization of lattice polynomial classes

General criterion (C. & Marichal)

Let C be a class of functions such that

(i) the unary members of C are polynomial functions;

(ii) any $g : X \to X$ obtained from $f\colon X^n \to X \in C$ by fixing $n-1$ arguments is in C.

Then C is a class of polynomial functions.

Extensions: pseudo-polynomial functions

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each X_i is a finite distributive lattice.

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Definition

We say that $f\colon \mathbf{X} \to X$ is a pseudo-polynomial function if

$$f(\mathbf{x}) = p\big(\varphi_1(x_1), \ldots, \varphi_n(x_n)\big),$$

where $p\colon X^n \to X$ is polynomial function and each $\varphi_i\colon X_i \to X$ satisfies

$$\varphi_i(0) \leq \varphi_i(x_i) \leq \varphi_i(1). \qquad \text{(BC)}$$

Fact: We can always choose p to be a Sugeno integral!

Sugeno utility functions as pseudo-polynomial functions

A function $f\colon \mathbf{X} \to X$ is a Sugeno utility function if

$$f(\mathbf{x}) = q\big(\varphi_1(x_1), \ldots, \varphi_n(x_n)\big),$$

where q is a Sugeno integral and each $\varphi_i\colon X_i \to X$ is order-preserving.

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Proposition (C. & Waldhauser)

Order-preserving pseudo-polynomial functions are Sugeno utility functions.

Problems...

Consider $f\colon \mathbf{X} \to X$.

Problem 1: Determine whether f is pseudo-polynomial function.

Problem 2: Find all possible factorizations $f = p\left(\varphi_1, \ldots, \varphi_n\right)$.

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Remark:

Problems 1 and 2 were solved (C. & Marichal) when $X_1 = \cdots = X_n$ and

$$f = p\left(\varphi(x_1), \ldots, \varphi(x_n)\right).$$

Such model is pertaining to **QDM under uncertainty**.

Properties of pseudo-polynomial functions (I)

We say that $f\colon \mathbf{X} \to X$ is pseudo-median decomposable if for each $i \in [n]$ there exists $\varphi_i\colon X_i \to X$ such that

$$f(\mathbf{x}) = \mathrm{median}\left(f(\mathbf{x}_i^0)\, ,\, \varphi_i(x_i)\, ,\, f(\mathbf{x}_i^1) \right), \quad \text{for all } \mathbf{x} \in \mathbf{X}.$$

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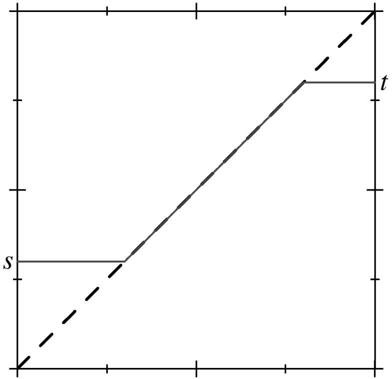

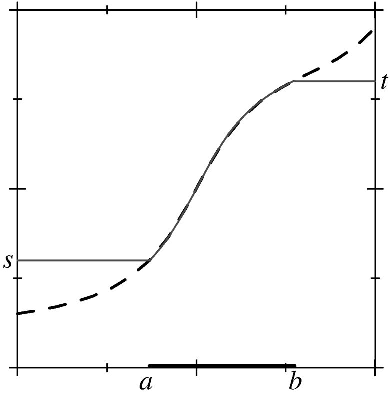

Characterizations of pseudo-polynomial functions (I)

Proposition (C. & Waldhauser)

If f is pseudo-median decomposable w.r.t. φ_i, **then** $f = p_f\,(\varphi_1, \ldots, \varphi_n)$

$$\textbf{where} \quad p_f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(f(\widehat{\mathbf{1}}_I) \wedge \bigwedge_{i \in I} x_i \Big).$$

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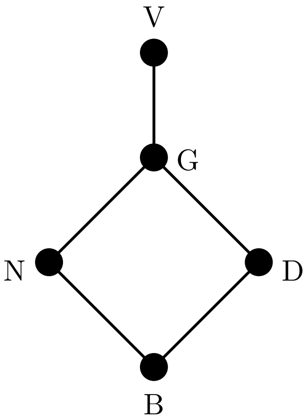
Theorem (C. & Waldhauser)

f is a pseudo-polynomial function **iff** it is pseudo-median decomposable.

Embedding a distributive lattice X into a power-set Y



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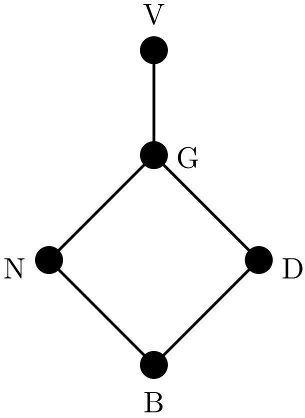
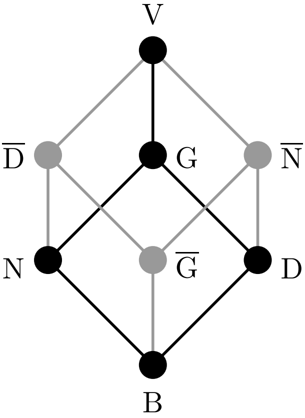
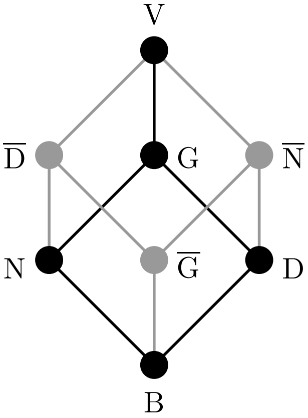




Closure and interior operators on Y



closure operator: $\mathrm{cl}\,(b) = \bigwedge_{\substack{a \in X \\ a \geq b}} a$

interior operator: $\mathrm{int}\,(b) = \bigvee_{\substack{a \in X \\ a \leq b}} a$

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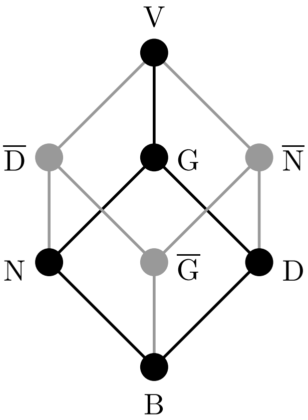

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$\mathsf{cl}\left(\overline{\mathrm{D}}\right) = \mathsf{cl}\left(\overline{\mathrm{N}}\right) = \mathsf{cl}\left(\overline{\mathrm{G}}\right) = \mathrm{V}$

$\mathsf{int}\left(\overline{\mathrm{D}}\right) = \mathrm{N}, \ \mathsf{int}\left(\overline{\mathrm{N}}\right) = \mathrm{D}, \ \mathsf{int}\left(\overline{\mathrm{G}}\right) = \mathrm{B}$

Towards necessary conditions...

Given $f\colon \mathbf{X} \to X$ and $i \in [n]$, define functions $\Phi_i^-, \Phi_i^+\colon X_i \to X$ by

$$\Phi_i^-(a_i) := \bigvee_{x_i = a_i} \mathrm{cl}\big(f(\mathbf{x}) \wedge \overline{f(\mathbf{x}_i^0)}\big),$$

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Proposition (C. & Waldhauser)

If $f\colon \mathbf{X} \to X$ is a pseudo-polynomial function, **then**

$$f = p_f(\varphi_1, \ldots, \varphi_n), \quad \text{for } \varphi_i \in \{\Phi_i^-, \Phi_i^+\}.$$

Characterization of pseudo-polynomial functions

Fact

If f is a pseudo-polynomial function, **then** it satisfies

$$f(\mathbf{x}_i^0) \leq f(\mathbf{x}) \leq f(\mathbf{x}_i^1). \qquad (\mathrm{BC}_n)$$

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Theorem (C. & Waldhauser)

The function f is a pseudo-polynomial function **iff**

1. f satisfies (BC_n)
2. for every $i \in [n]$, $\Phi_i^- \leq \Phi_i^+$.

When X is a finite chain

Theorem (C. & Waldhauser): For a finite chain X...

$f\colon \mathbf{X} \to X$ is pseudo-polynomial **iff** it satisfies (BC_n) and

$$f\left(\mathbf{x}_i^0\right) < f\left(\mathbf{x}_i^{a_i}\right) \text{ and } f\left(\mathbf{y}_i^{a_i}\right) < f\left(\mathbf{y}_i^1\right) \implies f\left(\mathbf{x}_i^{a_i}\right) \leq f\left(\mathbf{y}_i^{a_i}\right)$$

Finding the local utility functions

Theorem (C. & Waldhauser)

A function $\varphi_i \colon X_i \to X$ satisfying (BC) appears in a factorization of f **iff**

$$\Phi_i^- \leq \varphi_i \leq \Phi_i^+.$$

Finding all polynomial functions

Let $f\colon \mathbf{X} \to X$ and $\varphi_i\colon X_i \to X$ be given as before.

We define the polynomial functions $p^-, p^+\colon Y^n \to X$ by

$$p^-(\mathbf{y}) := \bigvee_{I \subseteq [n]} \big(c_I^- \wedge \bigwedge_{i \in I} x_i\big) \text{ with } c_I^- := \mathrm{cl}\Big(f(\widehat{\mathbf{1}}_I) \wedge \bigwedge_{i \notin I} \overline{\varphi_i(0)}\Big),$$

$$p^+(\mathbf{y}) := \bigvee_{I \subseteq [n]} \big(c_I^+ \wedge \bigwedge_{i \in I} x_i\big) \text{ with } c_I^+ := \mathrm{int}\Big(f(\widehat{\mathbf{1}}_I) \vee \bigvee_{i \in I} \overline{\varphi_i(1)}\Big).$$

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$$p^+(\mathbf{y}) := \bigvee_{I \subseteq [n]} \Big(c_I^+ \wedge \bigwedge_{i \in I} x_i\Big) \text{ with } c_I^+ := \mathrm{int}\Big(f(\widehat{\mathbf{1}}_I) \vee \bigvee_{i \in I} \overline{\varphi_i(1)}\Big).$$

Theorem (C. & Waldhauser)

For a polynomial function $p(\mathbf{y}) = \bigvee_{I \subseteq [n]} \Big(c_I \wedge \bigwedge_{i \in I} x_i\Big)$ we have

$f = p(\varphi_1, \ldots, \varphi_n)$ if and only if $c_I^- \leq c_I \leq c_I^+$ holds for all $I \subseteq [n]$.

Decision making **DM**

Main Problems

1. Model preference relations.
2. Axiomatize the chosen model.

Decision making **DM**

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1. Model preference relations.
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Question: What is a preference relation?

Preference relations

Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each X_i is a finite chain.

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Let $\mathbf{X} := X_1 \times \cdots \times X_n$, where each X_i is a finite chain.

A **weak order** on $\mathbf{X}$ is a relation $\preceq \subseteq \mathbf{X}^2$ that is:

1. reflexive: $\forall \mathbf{x} \in \mathbf{X} : \mathbf{x} \preceq \mathbf{x}$,
2. transitive: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X} : \mathbf{x} \preceq \mathbf{y},\ \mathbf{y} \preceq \mathbf{z} \implies \mathbf{x} \preceq \mathbf{z}$, and
3. complete: $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x} \preceq \mathbf{y}$ or $\mathbf{y} \preceq \mathbf{x}$.

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A **weak order** on $\mathbf{X}$ is a relation $\preceq \subseteq \mathbf{X}^2$ that is:

1. reflexive: $\forall \mathbf{x} \in \mathbf{X} : \ \mathbf{x} \preceq \mathbf{x}$,
2. transitive: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X} : \ \mathbf{x} \preceq \mathbf{y}, \ \mathbf{y} \preceq \mathbf{z} \implies \mathbf{x} \preceq \mathbf{z}$, and
3. complete: $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \ \mathbf{x} \preceq \mathbf{y}$ or $\mathbf{y} \preceq \mathbf{x}$.

Note: Weak orders are not necessarily antisymmetric:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \ \mathbf{x} \preceq \mathbf{y}, \ \mathbf{y} \preceq \mathbf{x} \implies \mathbf{x} = \mathbf{y} \tag{AS}$$

Indifference relation

The **indifference relation** $\sim$ associated with $\preceq$ is defined by:

$$\mathbf{y} \sim \mathbf{x} \;\; \textbf{iff} \;\; \mathbf{x} \preceq \mathbf{y} \;\; \text{and} \;\; \mathbf{y} \preceq \mathbf{x}.$$

Note that...

1. $\sim$ is an equivalence relation.
2. $\leq := \preceq / \sim$ satisfies (AS) and $\mathbf{X}/\sim$ is a (finite) chain.

Preference relations

A preference relation on $\mathbf{X}$ is a weak order $\preceq$ that satisfies

Pareto condition: $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \forall i \in [n], x_i \preceq_i y_i \implies \mathbf{x} \preceq \mathbf{y}$.

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Fact

The **rank function** $r \colon \mathbf{X} \to \mathbf{X}/\sim$ of $\preceq$ is order-preserving and:

$$\mathbf{x} \preceq \mathbf{y} \iff r(\mathbf{x}) \leq r(\mathbf{y}).$$

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Consequence:

Preference relations are exactly those representable by order-preserving functions.

Axiomatic approach to **QDM**

Model: Preference relations are represented by Sugeno utility functions.

Axiomatic approach to **QDM**

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Theorem (C. & Dubois & Waldhauser)

A relation $\preceq$ on $\mathbf{X}$ is representable by a Sugeno utility function **iff**

1. $\preceq$ is a preference relation
2. $\preceq$ satisfies: $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x}_i^0 \prec \mathbf{x}_i^a$ and $\mathbf{y}_i^a \prec \mathbf{y}_i^1 \implies \mathbf{x}_i^a \preceq \mathbf{y}_i^a$.

Proof

Theorem: For a finite chain X...

$f: \mathbf{X} \to X$ is a Sugeno utility function **iff** it is order-preserving and

$$f\left(\mathbf{x}_i^0\right) < f\left(\mathbf{x}_i^{a_i}\right) \text{ and } f\left(\mathbf{y}_i^{a_i}\right) < f\left(\mathbf{y}_i^1\right) \implies f\left(\mathbf{x}_i^{a_i}\right) \leq f\left(\mathbf{y}_i^{a_i}\right) \qquad (*)$$

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If $\preceq$ is a preference relation satisfying:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x}_i^0 \prec \mathbf{x}_i^a \text{ and } \mathbf{y}_i^a \prec \mathbf{y}_i^1 \implies \mathbf{x}_i^a \preceq \mathbf{y}_i^a,$$

then r is a Sugeno utility function representing $\preceq$.

Conversely...

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$f\colon \mathbf{X} \to X$ is a Sugeno utility function **iff** it is order-preserving and

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Conversely, suppose $\preceq$ is represented by a Sugeno utility function f.

Then we may assume that f is surjective.

Hence $r = \alpha \circ f$ for some order-isomorphism α.

Since f satisfies $(*)$, r satisfies $(*)$ and **thus**

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X} : \mathbf{x}_i^0 \prec \mathbf{x}_i^a \text{ and } \mathbf{y}_i^a \prec \mathbf{y}_i^1 \implies \mathbf{x}_i^a \preceq \mathbf{y}_i^a. \qquad \square$$

Remarks:

QDM under uncertainty: Single universe $X_0 = X_1 = X_2 = \cdots = X_n$ and a single utility function $\varphi \colon X_0 \to X$ for each $i \in [n]$.

1. Computational approach: Chateauneuf & Grabisch & Labreuche & Rico
2. Axiomatic treatment: Dubois & Fargier & Prade & Sabbadin

Further problems and directions of research:

1. Properties for aggregation (functional equations):

   Examples: associativity, commutation, scale invariance...

2. Aggregation on specific scales:

   Examples: ordinal, interval, bipolar scales...

3. Interpolation problems:

   Applications in AI: learning functions and preferences...

4. Fusion of (qualitative) information.

5. Construction methods.

6. ...

Thank you for your attention!