Idempotent tropical matrices

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Tropical matrices

Recall the **tropical semifield**, $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$, where

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b.$$

Let $M_n(\mathbb{FT})$ denote the set of all $n \times n$ matrices over $\mathbb{FT}$, with multiplication $\otimes$ defined as you would expect:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

It is easy to see that $(M_n(\mathbb{FT}), \otimes)$ is a **semigroup**.

Tropical polytopes

We write $\mathbb{FT}^n$ to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$. Then $\mathbb{FT}^n$ has the structure of an **$\mathbb{FT}$-module**:

$$(x \oplus y)_i = x_i \oplus y_i, \quad (\lambda \otimes x)_i = \lambda \otimes x_i,$$

for all $x, y \in \mathbb{FT}^n$ and all $\lambda \in \mathbb{FT}$.

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Let $A \in M_n(\mathbb{FT})$. We define the **row space** $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of A.

Similarly, we define the **column space** $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of A.

Dimensions of tropical polytopes

Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

- The **tropical dimension** of X is the maximum topological dimension of X regarded as a subset of $\mathbb{R}^n$.
- We say that X has **pure tropical dimension** k if every open subset of X has topological dimension k.

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In general, these dimensions can be different. However...

Idempotents, projectivity and dimensions

Theorem [IJK] Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

There is a positive integer k such that X has pure tropical dimension k, generator dimension k and dual dimension k

if and only if

X is the column space of an idempotent

if and only if

X is projective as an $\mathbb{FT}$-module.

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X is projective as an $\mathbb{FT}$-module.

- If this common dimension is k we say that X is a **projective k-polytope**.
- Moreover, if E is an idempotent with $X = C(E)$, we say that E has **rank k**. (Note: $1 \leqslant \mathrm{rank}(E) \leqslant n$.)

Maximal subgroups

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Theorem [IJK]

Let E be an idempotent in $M_n(\mathbb{FT})$ and let H_E denote the maximal subgroup containing E. Then

- H_E is isomorphic to the group of $\mathbb{FT}$-module automorphisms of the column space $C(E)$
- H_E is isomorphic to the group of $\mathbb{FT}$-module automorphisms of the row space $R(E)$.

Maximal subgroups for idempotents of full rank

Let $\mathbb{T} = \mathbb{FT} \cup \{-\infty\}$ and consider the monoid $M_n(\mathbb{T})$.
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Let E be an idempotent of rank n in $M_n(\mathbb{FT})$ and define

$$G_E = \{G : G \text{ is a unit in } M_n(\mathbb{T}) \text{ and } GE = EG\}.$$

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Theorem [IJK]
Every $\mathbb{FT}$-module automorphism of a projective n-polytope

(i) extends to an automorphism of $\mathbb{FT}^n$ and

(ii) is a (classical) affine linear map.

Maximal subgroups of $M_n(\mathbb{FT})$

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Let E be an idempotent of rank n n $M_n(\mathbb{FT})$.
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Corollary [IJK]

Let H be a maximal subgroup of $M_n(\mathbb{FT})$ containing a rank k idempotent. Then $H \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leqslant S_k$.

Idempotents, groups and finite metrics

Let $[n] = \{1, \ldots, n\}$ and let $d : [n] \times [n] \to \mathbb{R}$ be a metric.
Consider the $n \times n$ matrix E with $E_{i,j} = -d(i, j)$.
Then

- E is symmetric;
- $E \otimes E = E$;
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Theorem. [JK] Let E be an idempotent corresponding to a metric d on n points. Then $H_E \cong \mathbb{R} \times I$, where I is the isometry group of the finite metric space $([n], d)$.

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Corollary. **[JK]** Let G be a finite group. Then $\mathbb{R} \times G$ is a maximal subgroup of $M_n(\mathbb{FT})$, for n sufficiently large.