Piecewise testable languages and the word problem

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Definition

Let $w \in A^*$. *A word* $a_1 \dots a_l \in A^*$ *is a* *subword* *of* w *if* $\exists v_0, v_1, \dots, v_l \in A^*$, *such that* $w = v_0 a_1 v_1 a_2 \dots a_l v_l$.
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Let $u, v \in A^*$*,* $k \in \mathbb{N}$*.* $u \sim_k v \Leftrightarrow C_k(u) = C_k(v)$*.*

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- V_k: monoid variety corresponding to k-piecewise testable languages.
- F_{V_k}: the free monoid in V_k.

Classes of $\sim_k$

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$B_1 = B(a) \prec B(b) = B_2 \Rightarrow i_a = 1,\ i_b = i_c = 2.$

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$B_1 = \{a\}, \quad B_2 = \{b, c\} \Rightarrow i_a = 1,\ i_b = i_c = 2.$

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$\Rightarrow j_a = j_b = 2, \; j_c = 1.$

If a letter x occurs only once, then $j_x = i_x - 1$, otherwise $i_x \leq j_x$.

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4. The normal form $\overline{w} := w_1 v_1 w_2 \ldots w_t v_t$.

Example (cont. (cont.))

$w = acababa$

$i_a = 1,\ j_c = 1,\ i_b = i_c = 2,\ j_a = j_b = 2.$

$w_1 = a,\ \ j_c = 1$, but $2 = i_c \not\leq j_c \Rightarrow v_1 = \emptyset.$

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The normal form of $w \in F_{V_2}$

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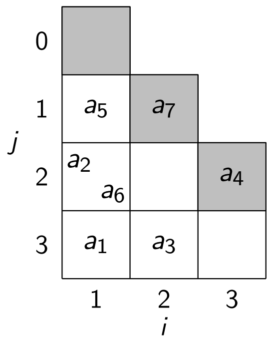
0
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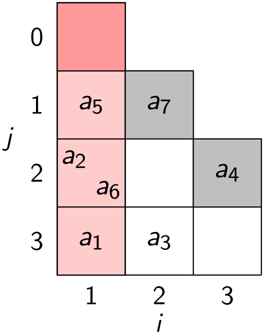
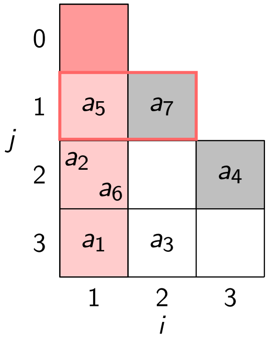


$\overline{w} = a_1 a_2 a_5 a_6$

w_1



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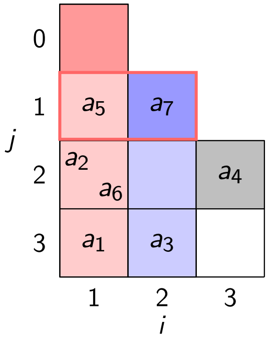
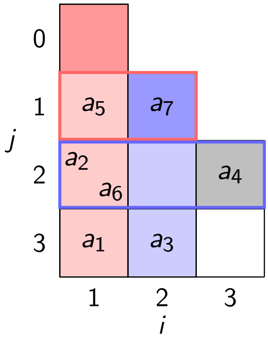
w_1 v_1



$$\overline{w} = a_1 a_2 a_5 a_6 a_5 a_7 a_3$$

w_1 v_1 w_2



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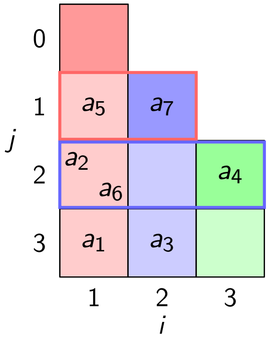
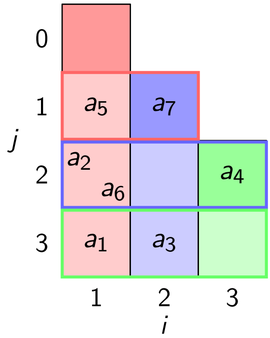


$$\overline{w} = a_1 a_2 a_5 a_6 a_5 a_7 a_3 a_2 a_6 a_4$$

w_1 v_1 w_2 v_2 w_3



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Note

In this step we eliminate a subword and replace it with a word containig only the letters of the block B.

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Free spectra

Theorem (K., Pach, Pluhár, Pongrácz, Szabó)

Let $|F_{V_k}(n)|$ denote the size of the n-generated free monoid in V_k

$$\log|F_{V_k}(n)| = \begin{cases} \Theta(n^{(k+1)/2}), & \text{if } k \text{ is odd,} \\ \Theta(n^{k/2}\log n), & \text{if } k \text{ is even,} \end{cases}$$

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where $\Theta(f) = g$ if $\exists d_1, d_2$ such that $d_1 \cdot g(n) \leq f(n) \leq d_2 \cdot g(n)$.