New Maximal Subsemigroups of the Semigroup of all Transformations on a countable set

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History

- 2002 L. Heindorf : The maximal clones on countable sets that include all permutations

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- April 2011 J. East, D. Mitchell, Y. Péresse: Maximal subsemigroups of the semigroup of all mappings on an infinite set containing all permutations
- September 2011 J. East, D. Mitchell, Y. Péresse: Maximal subsemigroups of the semigroup of all mappings on an infinite set.

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- the stabilizer of an ultrafilter on Ω.

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Problem

Characterization of all maximal subsemigroups of S

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For $M \subseteq \mathcal{P}(\Omega^\Omega)$, let $J(M)$ be the set of all $A \subseteq \bigcup M$ with $\forall m \in M(A \cap m \neq \emptyset) \& \forall a \in A \exists m \in M(A \cap m = \{a\})$

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Definition

For $U \subset \Omega^\Omega$ and $W \leq \Omega^\Omega$, we put
$Gen(U) := \{A \subseteq \Omega^\Omega \mid A \text{ is finite and } \langle A \rangle \cap U \neq \emptyset\}$ and
$\mathcal{H}(U, W) := \{A \subseteq \Omega^\Omega \setminus W \mid A \in J(Gen(U))\}$

Main theorem

Theorem

Let $W \leq S \leq \Omega^\Omega$ and $U \subset \Omega^\Omega$ such that each $\alpha \in U$ is a generator modulo W. Then the following statements are equivalent:
(i) S is maximal.
(ii) There is a set $H \in \mathcal{H}(U, W)$ with $S = \Omega^\Omega \setminus H$.

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Theorem

(L. Heindorf 2002)
Let $S \leq \Omega^\Omega$ containing the symmetric group. S is maximal iff $S = \Omega^\Omega \setminus H$ for some $H \in \{Inj(\Omega), Sur(\Omega), C_p(\Omega), IF(\Omega), FI(\Omega)\}$

Sur(X) and Inj(X)

- The maximal subsemigroups containing $Inj(\Omega)$ or $Sur(\Omega)$

Sur(X) and Inj(X)

- The maximal subsemigroups containing $Inj(\Omega)$ or $Sur(\Omega)$

Theorem

Let $S \leq \Omega^\Omega$ containing $Sur(\Omega)$. S is maximal iff $S = \Omega^\Omega \setminus Inj(\Omega)$ or $S = \Omega^\Omega \setminus FI(\Omega)$.

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Let $S \leq \Omega^\Omega$ containing $Inj(\Omega)$. S is maximal iff $S = \Omega^\Omega \setminus Sur(\Omega)$ or $S = \Omega^\Omega \setminus IF(\Omega)$ or $S = \Omega^\Omega \setminus C_p(\Omega)$.

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- maximal subsemigroups containing $FI(\Omega)$ (using main theorem)

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Lemma

Each $\alpha \in C_p(\Omega) \cap Sur(\Omega)$ is a generator modulo $FI(\Omega)$.

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Each $\alpha \in C_p(\Omega) \cap Sur(\Omega)$ is a generator modulo $FI(\Omega)$.

Theorem

Let $S \leq \Omega^\Omega$ containing $FI(\Omega)$. S is maximal iff $S = \Omega^\Omega \setminus H$ for some $H \in \mathcal{H}(C_p(\Omega) \cap Sur(\Omega), FI(\Omega))$.

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Each $\alpha \in FI(\Omega) \cap Inj(\Omega)$ is a generator modulo $IF(\Omega)$.

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Lemma

Each $\alpha \in FI(\Omega) \cap Inj(\Omega)$ is a generator modulo $IF(\Omega)$.

Theorem

Let $S \leq \Omega^\Omega$ containing $IF(\Omega)$. S is maximal iff $S = \Omega^\Omega \setminus H$ for some $H \in \mathcal{H}(Inj(\Omega) \cap FI(\Omega), IF(\Omega))$.

infinite contraction index

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Each $\alpha \in FI(\Omega) \cap Inj(\Omega)$ *is a generator modulo* $\langle C_p(\Omega) \rangle$.

Theorem

Let $S \leq \Omega^\Omega$ *containing* $IF(\Omega)$. S *is maximal iff* $S = \Omega^\Omega \setminus H$ *for some* $H \in \mathcal{H}(Inj(\Omega) \cap FI(\Omega), C_p(\Omega))$.