Generalized entropy in algebras with neutral element and in inverse semigroups

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Commuting operations

Let A be an arbitrary set, and n and m positive integers.

We denote $[n] := \{1, \dots, n\}$.

Definition

We say that $f\colon A^n \to A$ and $g\colon A^m \to A$ **commute** if

$$g\big(f(a_{11}, a_{12}, \dots, a_{1n}), \dots, f(a_{m1}, a_{m2}, \dots, a_{mn})\big) = f\big(g(a_{11}, a_{21}, \dots, a_{m1}), \dots, g(a_{1n}, a_{2n}, \dots, a_{mn})\big),$$

for all $a_{ij} \in A$ ($i \in [m]$, $j \in [n]$).

If f and g commute, then we write $f \perp g$.

Commuting operations

In other words, f and g commute if

$$
\begin{array}{cccccccc}
 & \overset{f}{\frown} & \overset{f}{\frown} & & \overset{f}{\frown} & & & \overset{f}{\frown} \\
g(& a_{11} & a_{12} & \cdots & a_{1m} &) & = & c_1 \\
g(& a_{21} & a_{22} & \cdots & a_{2m} &) & = & c_2 \\
 & \vdots & \vdots & & \vdots & & & \vdots \\
g(& a_{n1} & a_{n2} & \cdots & a_{nm} &) & = & c_n \\
 & \smile & \smile & & \smile & & & \smile \\
 & \parallel & \parallel & & \parallel & & & \parallel \\
g(& d_1 & d_2 & \cdots & d_m &) & = & b
\end{array}
$$

Entropic algebras

Definition

An algebra $\mathbf{A} = (A; F)$ is called entropic if every pair of its fundamental operations commute.

Generalized entropy

Definition

An algebra $\mathbf{A} = (A; F)$ has the generalized entropic property if, for every n-ary $f \in F$ and every m-ary $g \in F$, there exist m-ary terms $t_1, \dots, t_n$ of $\mathbf{A}$ such that $\mathbf{A}$ satisfies the identity

$$g\big(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})\big) \approx f\big(t_1(x_{11}, \dots, x_{1m}), \dots, t_n(x_{n1}, \dots, x_{nm})\big).$$

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Remark

Entropy implies generalized entropy.

Examples of entropic algebras and of algebras with the generalized entropic property

- Every commutative semigroup is entropic.
- There are non-commutative semigroups that are entropic, e.g., any left-zero band (a groupoid satisfying $xy \approx x$).
- The variety of groupoids satisfying

$$(x_1 x_2)(x_3 x_4) \approx (x_3 x_1)(x_2 x_4)$$

  has the generalized entropic property but it is not entropic (Adaricheva, Pilitowska, Stanovský (2008)). Thus, there exist non-commutative semigroups that have the generalized entropic property but are not entropic.

Subalgebras property and generalized entropy

Definition

An algebra $\mathbf{A} = (A; F)$ is said to have the subalgebras property if, for each n-ary operation $f \in F$, the complex product

$$f(A_1, \ldots, A_n) := \{f(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$$

of its (nonempty) subalgebras $A_1, \ldots, A_n$ is again a subalgebra.

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Theorem (Adaricheva, Pilitowska, Stanovský (2008))

Let $\mathcal{V}$ be a variety of algebras. Then each algebra in $\mathcal{V}$ has the subalgebras property if and only if each algebra in $\mathcal{V}$ has the generalized entropic property.

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N.B. For an algebra, the subalgebras property does not necessarily imply generalized entropy.

Neutral elements

Definition

An element $e \in A$ is neutral for an operation $f\colon A^n \to A$, if

$$f(a, e, \ldots, e) = f(e, a, e, \ldots, e) = \cdots = f(e, \ldots, e, a) = a$$

for every $a \in A$.

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Definition

An element $e \in A$ is neutral for an algebra $(A; F)$ if e is neutral for each operation $f \in F$.

- Every $e \in A$ is neutral for the identity map on A; this is the only unary operation that has a neutral element.
- Nullary operations do not have neutral elements.
- If e is neutral for an algebra $(A; F)$, then $\{e\}$ is a subalgebra of $(A; F)$.

Generalized entropy in algebras with a neutral element

Theorem (Adaricheva, Pilitowska, Stanovský (2008))

Let $(A; F)$ be an algebra with a neutral element. Then $(A; F)$ has the generalized entropic property if and only if it is entropic.

Generalized entropy in algebras with a neutral element

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Theorem

Let $\mathbf{A} = (A; F)$ be an algebra of type τ with a neutral element e. Then $\mathbf{A}$ has the generalized entropic property (or, equivalently, $\mathbf{A}$ is entropic), if and only if there exists a commutative monoid $(A; f, e)$ such that $\mathbf{A}$ is the τ-algebra derived from f.

Derived algebras

Let $f\colon A^n \to A$, $n \geq 1$. For $\ell \geq 0$, define the operation $f^{(\ell)}$ of arity $N(\ell) := \ell(n-1)+1$ recursively as

- $f^{(0)} := \mathrm{id}_A$,
- for $\ell \geq 0$, let

$$f^{(\ell+1)}(a_1, \dots, a_{N(\ell+1)}) = f(f^{(\ell)}(a_1, \dots, a_{N(\ell)}), a_{N(\ell)+1}, \dots, a_{N(\ell+1)}),$$

  for all $a_1, \dots, a_{N(\ell+1)} \in A$.

Note that $f^{(1)} = f$.

Derived algebras

Definition

An algebra $(A; (f_i)_{i \in I})$ of type $\tau = (n_i)_{i \in I}$ is the τ-algebra derived from f, if for every $i \in I$, there exists an integer $\ell_i \geq 0$ such that $n_i = N(\ell_i)$ and $f_i = f^{(\ell_i)}$.

Inverse semigroups

An inverse semigroup is an algebra $\mathbf{A} = (A; \cdot, {}^{-1})$ of type $(2, 1)$ that satisfies the following identities:

- $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$ (associativity),
- $x \cdot x^{-1} \cdot x \approx x$,
- $x^{-1} \cdot x \cdot x^{-1} \approx x^{-1}$.

Inverse semigroups

Facts about inverse semigroups:

- Idempotents commute.
- Elements of the form xx^{-1} and $x^{-1}x$ are idempotent.
- $(x^{-1})^{-1} \approx x$.
- $(xy)^{-1} \approx y^{-1}x^{-1}$.
- $x^k x^{-k} x^k \approx x^k$ and $x^{-k} x^k x^{-k} \approx x^{-k}$ for any natural number k.

Entropic inverse semigroups

Theorem

An inverse semigroup is entropic if and only if it is commutative.

Proof.

Assume first that $\mathbf{A} = (A; \cdot, ^{-1})$ is commutative. We have:

- $\cdot \perp \cdot$, because $(xy) \cdot (uv) \approx (xu) \cdot (yv)$;
- $\cdot \perp {}^{-1}$, because $(xy)^{-1} \approx (yx)^{-1} \approx x^{-1}y^{-1}$;
- ${}^{-1} \perp {}^{-1}$, trivially.

We conclude that $\mathbf{A}$ is entropic.

Assume then that $\mathbf{A}$ is entropic. Then $(xy)^{-1} \approx x^{-1}y^{-1}$. On the other hand, we have $(xy)^{-1} \approx y^{-1}x^{-1}$. This implies that $xy \approx yx$, i.e., $\mathbf{A}$ is commutative. □

Inverse semigroups with generalized entropic property

Theorem

An inverse semigroup has the generalized entropic property if and only if it is commutative.

In order to prove this theorem, we made use of the following representation of the free monogenic inverse semigroup, which has been attributed to Schein and to Gluskin. Each member of the free monogenic inverse semigroup has a canonical form

$$x^{-p}x^{q}x^{-r},$$

where $0 \leq p \leq q$, $0 \leq r \leq q$, $q > 0$. (Convention: x^0 is an empty symbol.) The canonical form of the product

$$(x^{-p_1}x^{q_1}x^{-r_1})(x^{-p_2}x^{q_2}x^{-r_2})$$

is $x^{-p}x^{q}x^{-r}$, where

$$\begin{aligned}
p &= p_1 + r_1 + p_2 - \min\{q_1, r_1 + p_2\}, \\
q &= q_1 + r_1 + p_2 + q_2 - \min\{q_1, r_1 + p_2\} + \min\{q_2, r_1 + p_2\}, \\
r &= r_1 + p_2 + r_2 - \min\{q_2, r_1 + p_2\}.
\end{aligned}$$

Inverse semigroups with generalized entropic property

Lemma

Let **A** *be an inverse semigroup and let*

$$t_1(x) = x^{-p_1} x^{q_1} x^{-q_1}, \quad t_2(y) = y^{-p_2} y^{q_2} y^{-r_2}$$

for some $0 \leq p_1 \leq q_1 \neq 0$, $0 \leq r_1 \leq q_1$, $0 \leq p_2 \leq q_2 \neq 0$, $0 \leq r_2 \leq q_2$. *Assume that* **A** *satisfies the identity*

$$(xy)^{-1} \approx t_1(x) t_2(y).$$

Then there exist positive integers a *and* b *such that* **A** *satisfies the identity* $xy \approx y^b x^a$.

Inverse semigroups with generalized entropic property

Proof.

The proof is a lengthy case analysis, according to whether strict inequality or equality holds in each of the inequalities involving p_1, q_1, r_1, p_2, q_2, r_2.

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For example, consider the case when $q_1 = r_1$ and $p_1 = 0$, i.e., $t_1(x) = x^{q_1} x^{-q_1}$. By the idempotency of $x^{-1}x$ we obtain

$$x^{-1} \approx (x(x^{-1}x))^{-1} \approx t_1(x) t_2(x^{-1}x) \approx x^{q_1} x^{-q_1} x^{-1} x.$$

Since $x^{q_1} x^{-q_1}$ is an idempotent, too, this implies that x^{-1} is a product of idempotents and is hence itself an idempotent. Thus every element of **A** is idempotent. Since idempotents of an inverse semigroup commute, this implies that **A** is commutative, i.e., **A** satisfies $xy \approx yx$.

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Several other cases . . . $\square$

Inverse semigroups with generalized entropic property

Lemma

Let **A** *be an inverse semigroup that satisfies* $xy \approx y^b x^a$ *for some positive integers* a *and* b*. Then* **A** *satisfies:*

1. $x^{-1}x^{a+1} \approx x \approx x^{b+1}x^{-1}$,
2. $x^{a+1} \approx x^2 \approx x^{b+1}$,
3. $x^{-1}x^2 \approx x \approx x^2x^{-1}$.

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Proof.

1. $x \approx x(x^{-1}x) \approx (x^{-1}x)^b x^a \approx x^{-1}xx^a \approx x^{-1}x^{a+1}$,
   $x \approx (xx^{-1})x \approx x^b(xx^{-1})^a \approx x^b xx^{-1} \approx x^{b+1}x^{-1}$.
2. $x^2 \approx xx^{-1}x^{a+1} \approx x^{a+1}$,
   $x^2 \approx x^{b+1}x^{-1}x \approx x^{b+1}$.
3. Follows immediately from (1) and (2). □

Inverse semigroups with generalized entropic property

Corollary

Let $\mathbf{A}$ *be an inverse semigroup that satisfies* $xy \approx y^b x^a$ *for some positive integers* a *and* b*. Then* $\mathbf{A}$ *is commutative.*

Inverse semigroups with generalized entropic property

Corollary

Let **A** *be an inverse semigroup that satisfies* $xy \approx y^b x^a$ *for some positive integers* a *and* b. *Then* **A** *is commutative.*

Proof.

$xy \approx y^b x^a \approx y^{b-1} yxx^{a-1} \approx y^{b-1} y^2 y^{-1} x^{-1} x^2 x^{a-1} \approx y^{b+1} y^{-1} x^{-1} x^{a+1} \approx yx.$ □

Thank you for your attention!