Properties of the automorphism group and a probabilistic construction of a class of countable labeled structures

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Automorphism groups of Fraïssé limits

Questions:

- Probabilistic construction
- Simplicity of the automorphism group
- Small index property
- Bergman property

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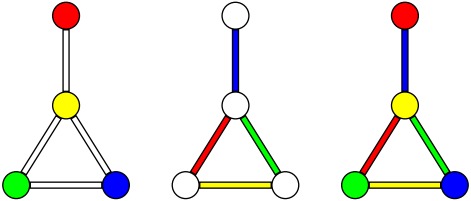
A helpful assumption: Aut($\mathcal{F}$) is oligomorphic

Contrast: rational Urysohn space

Automorphism groups of Fraïssé limits

We would like to consider some of these questions **but** in the setting where Aut($\mathcal{F}$) is not oligomorphic.

Our starting point: labeled graphs



The requirement that Aut($\mathcal{F}$) be oligomorphic will be replaced by other types finiteness requirements.

Labeled structures

$L = \{R_i : i \in \mathbb{N}\}$ — countable relational language

$\mathrm{Ar}(L) = \{\mathrm{ar}(R) : R \in L\}$

A relational language L has *bounded arity* if there is an $n \in \mathbb{N}$ such that $\mathrm{Ar}(L) \subseteq \{1, 2, \ldots, n\}$.

Definition. An L-structure A is *labeled* if for every $n \in \mathrm{Ar}(L)$ and every $\overline{a} \in A^n$ there exists *exactly one* $R \in L_n$ such that $A \models R(\overline{a})$.

An L-structure A is *partially labeled* if for every $n \in \mathrm{Ar}(L)$ and every $\overline{a} \in A^n$ there exists *at most one* $R \in L_n$ such that $A \models R(\overline{a})$.

Labeled structures

Definition. A labeled L-structure A^* is a *filling* of a partially labeled L-structure A if they have the same base set and $A \preccurlyeq A^*$.

A class $\mathcal{A}$ of partially labeled L-structures *has uniform fillings in a class $\mathcal{B}$ of labeled L-structures* if there is a mapping

$$(\cdot)^* : \mathcal{A} \to \mathcal{B}$$

such that for all $A, B \in \mathcal{A}$:

- A^* is a filling of A, and
- if f is an isomorphism from A onto B, then f is also an isomorphism from A^* onto B^*.

Our labeled structures may implement certain restrictions expressed by means of special Horn clauses over $L \cup \{\neq\}$.

A *Horn restriction over L* is a Horn clause of the form

$$\Phi = \neg(R_1(\overline{v}_1) \wedge \ldots \wedge R_n(\overline{v}_n))$$

where $R_1, \ldots, R_n \in L \cup \{\neq\}$ and $R_i \in L$ for at least one i.

Labeled structures – The setup

Condition (A)

- L — a countable relational language of bounded arity
- Σ — a set of Horn restrictions over L
- $\Sigma|_{L_0}$ is finite for every finite $L_0 \subseteq L$
- $\mathcal{P}_\Sigma$ — the class of all finite partially lbld L-structures satisfying Σ
- $\mathcal{K}_\Sigma$ — the class of all labeled structures in $\mathcal{P}_\Sigma$
- for all $A, B \in \mathcal{K}_\Sigma$: $A \sqcup B \in \mathcal{P}_\Sigma$
- for all $A, B, C \in \mathcal{K}_\Sigma$: $B \sqcup_A C$ belongs to $\mathcal{P}_\Sigma$;
- $\mathcal{P}_\Sigma$ has uniform fillings in $\mathcal{K}_\Sigma$.

Fact. $\mathcal{K}_\Sigma$ is a Fraïssé class.

Example: Graphs

$L = \{R_0(\cdot,\cdot), R_1(\cdot,\cdot)\}$

$$\Sigma: \quad \neg R_1(x,x)$$
$$\neg(R_1(x,y) \wedge R_0(y,x) \wedge x \neq y)$$

Uniform fillings:

- if (a,a) is not labeled in G, label it by R_0;
- if $R_1(a,b)$ but (b,a) is unlabeled, label (b,a) by R_1;
- if neither (a,b) nor (b,a) are labeled, label both by R_0.

Example: K_m-free graphs

$L = \{R_0(\cdot,\cdot), R_1(\cdot,\cdot)\}$

$$\begin{aligned}\Sigma : \quad & \neg R_1(x,x) \\ & \neg(R_1(x,y) \wedge R_0(y,x) \wedge x \neq y) \\ & \neg\left(\bigwedge_{1 \leqslant i < j \leqslant m} \left(x_i \neq x_j \wedge R_1(x_i,x_j) \wedge R_1(x_j,x_i)\right)\right)\end{aligned}$$

Uniform fillings:

- if (a,a) is not labeled in G, label it by R_0;
- if $R_1(a,b)$ but (b,a) is unlabeled, label (b,a) by R_1;
- if neither (a,b) nor (b,a) are labeled, label both by R_0.

Example: Edge-colored graphs

I — a nonempty countable set
$0 \in I$ — arbitrary but fixed

$L_I = \{R_b(\cdot, \cdot) : b \in I\}$

Σ_I : $\neg R_b(x, x)$ for all $b \in I \setminus \{0\}$
$\neg(R_b(x, y) \wedge R_c(y, x) \wedge x \neq y)$ for all $b, c \in I$ s. t. $b \neq c$

Uniform fillings:
symmetrize, label every unlabeled tuple by R_0.

Example: Metric spaces with rational distances

$L_{met} = \{D_q : q \in \mathbb{Q}^{\geqslant 0}\}$

Σ_{met} : $\neg(x \neq y \wedge D_0(x, y))$
$\neg D_q(x, x)$ for every $q \in \mathbb{Q}$ s. t. $q > 0$
$\neg(D_p(x, y) \wedge D_q(y, x))$ for all $p, q \in \mathbb{Q}^{\geqslant 0}$ s. t. $p \neq q$
$\neg(D_{q_1}(u_1, v_1) \wedge \ldots \wedge D_{q_n}(u_n, v_n) \wedge D_{q_0}(u_0, v_0)),$
for all $q_0, q_1, \ldots, q_n \in \mathbb{Q}$ s. t. $q_0, q_1, \ldots, q_n > 0$
and $q_0 > q_1 + \ldots + q_n,$ and all possible choices
$(u_i, v_i) \in \{(x_{i-1}, x_i), (x_i, x_{i-1})\}$ where $1 \leq i \leq n$
and $(u_0, v_0) \in \{(x_0, x_n), (x_n, x_0)\}$

Uniform fillings: nontrivial, but obvious

A negative example

$(\mathbb{Q}, <)$

[up to 1-dim bi-interpretability]

The small index property

$G = \text{Aut}(\mathcal{F})$ has the small index property if, for every $H \leqslant G$:

$$H \text{ is open if and only if } (G : H) < 2^\omega$$

Theorem. Assume that (A) holds and let K_Σ be the Fraïssé limit of $\mathcal{K}_\Sigma$. Then K_Σ has ample generic automorphisms, and therefore it has the small index property.

cf. A. S. Kechris, C. Rosendal: Turbulence, amalgamation, and generic automorphisms of homogeneous structures, Proc. London Math. Soc. (3) 94 (2007) 302–350.

The small index property

Consequently, the following Fraïssé limits have the small index property:

- the random graph $\mathcal{R}$ (proved by W. Hodges, I. Hodkinson, D. Lascar, S. Shelah 1993),
- the Henson graph $\mathcal{H}_m$, $m \geqslant 3$ (proved by Herwig 1998),
- the edge-colored random graph over a countable set of colors I (if I is finite, the strong small index property was proved by Cameron and Tarzi),
- the random deterministic transition system over a countable set of transitions I,
- the random I-fuzzy graph, where I is a countable bounded meet-semilattice,
- the rational Urysohn space, and the Urysohn sphere of radius 1 (follows from the results of Kechris and Rosendal 2007, Solecki 2005).

The Bergman property

An infinite group G has the *Bergman property* if for any generating subset $E \subseteq G$ such that $1 \in E = E^{-1}$ we have $G = E^k$ for some positive integer k.

Droste, Göbel 2005:
strong uncountable cofinality $\Rightarrow$ Bergman property

The Bergman property

Condition (A+) Assume that (A) holds, and that there is a uniform filling $(\cdot)^* : \mathcal{P}_\Sigma \to \mathcal{K}_\Sigma$ such that:

- for all $A, B, C, D \in \mathcal{K}_\Sigma$ such that $C \cap D = \varnothing$, if $f : C \hookrightarrow A$ and $g : D \hookrightarrow B$, then $f \cup g : (C \sqcup D)^* \hookrightarrow (A \sqcup B)^*$; and
- for all pairwise disjoint $A, B, C \in \mathcal{K}_\Sigma$ we have $((A \sqcup B)^* \sqcup C)^* = (A \sqcup (B \sqcup C)^*)^*$.

Example. Metric spaces with rational distances do not fulfill (A+), but if the distances are bounded by 1, then (A+) holds.

The Bergman property

Theorem. Assume that (A+) holds and let K_Σ be the Fraïssé limit of $\mathcal{K}_\Sigma$. Then Aut(K_Σ) has [strong uncountable cofinality, and consequently] the Bergman property.

cf. C. Rosendal: A topological version of the Bergman property, Forum Math. 21 (2009) 299–332.

The Bergman property

The automorphism groups of the following Fraïssé limits have the Bergman property:

- the random graph $\mathcal{R}$ (Kechris and Rosendal 2007),
- the Henson graph $\mathcal{H}_m$, $m \geqslant 3$ (Kechris and Rosendal 2007),
- the edge-colored random graph over a cntbl set of colors I,
- the random deterministic transition system over a countable set of transitions I,
- the random I-fuzzy graph, where I is a countable bounded meet-semilattice, and
- the Urysohn sphere of radius 1 (Rosendal 2009).

A general probabilistic construction

Recall: $\mathcal{K}_\Sigma$ is a Fraïssé class, so let K_Σ be its Fraïssé limit.

$\mu_n(\cdot)$ — prob measure on L_n s. t. $\mu_n(R) > 0$ for all $R \in L_n$

We start with $\Phi_0 = \varnothing \in \mathcal{P}_\Sigma$

Given a labeled L-structure $\Phi_n \in \mathcal{P}_\Sigma$ with the base set $\{a_1, \ldots, a_n\}$ we construct $\Phi_{n+1} \in \mathcal{K}_\Sigma \subseteq \mathcal{P}_\Sigma$ with the base set $\{a_1, \ldots, a_n, a_{n+1}\}$ as follows.

A general probabilistic construction

Step 1. Choose a new point $a_{n+1} \notin \Phi_n$, let $\Phi_{n+1}^{(0)} = \Phi_n \cup \{a_{n+1}\}$ where a_{n+1} is an isolated point. NB: $\Phi_{n+1}^{(0)} \in \mathcal{P}_\Sigma$.

Step 2. Arrange all admissible words over the alphabet $\{a_1, \ldots, a_{n+1}\}$ in a cunning manner:

$$\underbrace{\overline{a}_1, \ldots, \overline{a}_{l_1}}_{w_0}, \underbrace{\overline{a}_{l_1+1}, \ldots, \overline{a}_{l_2}}_{w_1}, \ldots, \underbrace{\overline{a}_{l_n+1}, \ldots, \overline{a}_m}_{w_n}$$

A general probabilistic construction

Step 3. For each $j \in \{0, \dots, m-1\}$ we take $\Phi_{n+1}^{(j)}$ and construct $\Phi_{n+1}^{(j+1)}$ on the same base set as follows.

Let k be the length of the tuple $\overline{a}_{j+1}$, which is a tuple that contains a_{n+1} and is not labeled in $\Phi_{n+1}^{(j)}$. Let

$$M_{j+1} = \{R \in L_k \ : \ \Phi_{n+1}^{(j)}\langle R, \overline{a}_{j+1}\rangle \preccurlyeq \Psi \text{ for some } \Psi \in \mathcal{K}_\Sigma \text{ with the same base set as } \Phi_{n+1}^{(j)}\}.$$

Clearly, $M_{j+1} \neq \varnothing$, so choose $R \in M_{j+1}$ with probability $\mu_k(R \mid M_{j+1})$ and let $\Phi_{n+1}^{(j+1)} = \Phi_{n+1}^{(j)}\langle R, \overline{a}_{j+1}\rangle$.

A general probabilistic construction

Step 4. By the construction, $\Phi_{n+1}^{(m)}$ is a labeled structure. Therefore, $\Phi_{n+1}^{(m)} \in \mathcal{K}_\Sigma$, and we set $\Phi_{n+1} = \Phi_{n+1}^{(m)}$.

Then $\Phi_1 \leqslant \Phi_2 \leqslant \ldots$ is an increasing chain of *L*-structures from $\mathcal{K}_\Sigma$ whose limit $\Phi = \bigcup_{i \geqslant 1} \Phi_i$ is K_Σ.

A general probabilistic construction

Vershik – a probabilistic construction of the rational Urysohn space.

Vershik constructs successive one-point extensions of finite metric spaces, using probabilities assigned to entire one-point extensions. These probability measures can be seen as “outer” measures since they take into account complete structures.

We can specialize the above general probabilistic construction to metric spaces and thus obtain an “inner” probabilistic construction of the rational Urysohn space which is more in the fashion of Erdős and Rényi’s probabilistic construction of R.