Inverse Semigroups: some open questions

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Six Problems

Problem 1: Does every finite inverse semigroup admit a finite F-inverse cover?

Problem 2: Is the word problem decidable for all one-relation semigroups?

Problem 3: Is the word problem decidable for one-relator monoids of the form $M = Inv\langle X : w = 1\rangle$ where w is cyclically reduced?

Problem 4: Is the prefix membership problem decidable for all cyclically reduced words?

Problem 5: When is the word problem for amalgamated free products of inverse semigroups decidable?

Problem 6: Is the consistency problem for single variable equations in free inverse monoids decidable?

Inverse semigroups

An inverse semigroup is a semigroup S with the property that for each $a \in S$ there is a unique element $a^{-1} \in S$ such that

$$a = aa^{-1}a \text{ and } a^{-1} = a^{-1}aa^{-1}$$

Inverse semigroups may be viewed as semigroups of partial injections between subsets of a set (or partial isometries of a Hilbert space or isomorphisms between subalgebras of some algebraic structure etc).

They may be characterized as regular semigroups whose idempotents commute. The set $E(S)$ of idempotents of an inverse semigroup S forms a (lower) semilattice with respect to $e \wedge f = ef$ for all $e, f \in E(S)$.

An inverse semigroup S comes equipped with a natural partial order defined by $a \leq b$ iff there exists $e \in E(S)$ such that $a = eb$.

Free Inverse Monoids

Inverse monoids form variety of algebras defined by the identities:

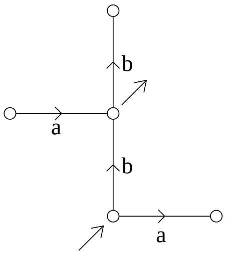
$$1a = a1 = a; (ab)c = a(bc); aa^{-1}a = a; (a^{-1})^{-1} = a;$$
$$(ab)^{-1} = b^{-1}a^{-1}, aa^{-1}bb^{-1} = bb^{-1}aa^{-1}$$

It follows that free inverse semigroups (monoids) exist. We will denote the free inverse monoid on a set X by $FIM(X)$. Thus $FIM(X)$ is the quotient of the free monoid $(X \cup X^{-1})^*$ obtained by factoring out the congruence generated by the identities above.

The structure of free inverse monoids was determined independently by Scheiblich and Munn in the 1970's. Scheiblich's approach was a precursor to McAlister's beautiful work on E-unitary inverse semigroups and Munn's approach lends itself to a pleasant solution to the word problem for $FIM(X)$ and is a precursor to the theory of presentations of inverse monoids by generators and relations.

Munn trees

Denote by $FG(X)$ the free group on X and by $\Gamma(X)$ the Cayley graph of $(FG(X), \emptyset)$. For each word $w \in (X \cup X^{-1})^*$, denote by $MT(w)$ the finite subtree of the tree $\Gamma(X)$ obtained by reading the word w as the label of a path in $\Gamma(X)$, starting at 1. Thus, for example, if $w = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$, then $MT(w)$ is the tree pictured below.



The word problem for $FIM(X)$

One may view $MT(w)$ as a birooted tree, with initial root 1 and terminal root $r(w)$, the reduced form of the word w in the usual group-theoretic sense. Munn's solution to the word problem in $FIM(X)$ may be stated in the following form.

Theorem

(Munn) If $u, v \in (X \cup X^{-1})^*$, then $u = v$ in $FIM(X)$ iff $MT(u) = MT(v)$ and $r(u) = r(v)$.

Elements of $FIM(X)$ may be viewed as pairs $(MT(w), r(w))$. Multiplication in $FIM(X)$ is performed as follows. If $u, v \in (X \cup X^{-1})^*$, then $MT(uv) = MT(u) \cup r(u).MT(v)$.

An equivalent description of $FIM(X)$ in terms of Schreier subsets of $FG(X)$ was provided independently by Scheiblich.

Basic structure of free inverse monoids

- The idempotents of $FIM(X)$ are the Dyck words, i.e. the words in $(X \cup X^{-1})^*$ whose reduced form is 1 (and two such words represent the same idempotent in $FIM(X)$ iff they span the same Munn tree).
- The Green's relations $\mathcal{D}$ and $\mathcal{J}$ on $FIM(X)$ coincide, and all $\mathcal{D}$-classes are finite
- the maximal subgroups of $FIM(X)$ are trivial.
- There is a natural homomorphism $(MT(w), r(w)) \to r(w)$ from $FIM(X)$ onto $FG(X)$, the maximal group homomorphic image of $FIM(X)$. This map is idempotent-pure, i.e. the inverse image of the identity of $FG(X)$ is precisely the semilattice of idempotents of $FIM(X)$.

E-unitary inverse semigroups

An inverse semigroup S that admits an idempotent-pure homomorphism onto a group is referred to as an *E-unitary* inverse semigroup. Thus $FIM(X)$ is an E-unitary inverse monoid.

Every inverse semigroup S admits a smallest congruence σ such that S/σ is a group: in fact σ is defined by $a \, \sigma \, b$ iff there exists $c \in S$ such that $c \leq a, b$. Thus S is E-unitary iff the semilattice $E(S)$ is a σ-class.

A theorem of McAlister states that every inverse semigroup T admits an *E-unitary cover*. That is, there exists an E-unitary inverse semigroup S and a surjective homomorphism $\phi : S \to T$ such that if e and f are distinct idempotents of S then $\phi(e) \neq \phi(f)$ in T. If S has maximal group image G then we say that T admits an E-unitary cover over G.

The structure of E-unitary inverse semigroups is determined by McAlister's "P-theorem". See the book by Lawson for much detail.

Problem 1: F-inverse monoids

An inverse monoid S is called F-inverse if each σ-class of S contains a maximal element.

Every F-inverse monoid is E-unitary, but the converse is false.

The free inverse monoid $FIM(X)$ is F-inverse.

It is known that every inverse semigroup admits an F-inverse cover. However, if S is finite, the proof of this produces an infinite F-inverse cover.

Problem 1: Does every finite inverse semigroup admit a finite F-inverse cover?

Presentations of inverse monoids:basic concepts

The inverse monoid M presented by a set X of generators and relations of the form $u_i = v_i$ will be denoted by $M = Inv\langle X : u_i = v_i \rangle$. Here $u_i, v_i \in (X \cup X^{-1})^*$. M is the image of the free inverse monoid $FIM(X)$ by the congruence generated by the defining relations, or equivalently, it is the quotient of $(X \cup X^{-1})^*$ obtained by applying the relations $u_i = v_i$ and the identities that define the variety of inverse monoids.

It is easy to see that $G = Gp\langle X : u_i = v_i \rangle$ is the maximal group homomorphic image of the inverse monoid $M = Inv\langle X : u_i = v_i \rangle$. For example, the bicyclic monoid has a presentation $Inv\langle a : aa^{-1} = 1 \rangle$ as an inverse monoid (and its maximal group image is $\mathbb{Z}$). Of course $FIM(X) = Inv\langle X : \emptyset \rangle$ and its maximal group image is $FG(X)$.

Problem 2: One-relation Semigroups

Magnus proved in the 1930's that every group with a single defining relation has decidable word problem. The corresponding problem for semigroups with a single defining relation remains unsolved.

Problem 2: Does every one-relation semigroup with a presentation of the form $S = Sgp\langle X : u = v \rangle$ (where $u, v \in X^*$) have decidable word problem?

There is considerable literature on this problem. Adian showed that the word problem for such a monoid is decidable if $v = 1$, i.e. if the semigroup is a monoid with presentation $S = Mon\langle X : w = 1 \rangle$ for some word $w \in X^+$.

The general problem remains open. A result of Adian and Oganessian reduces the word problem for one relation semigroups to the study of the word problem for semigroups with a presentation of the form $S = Sgp\langle X : aub = avc \rangle$ where $a, b, c \in X, b \neq c$ and $u, v \in X^*$.

Problem 3: One-relator inverse monoids

The word problem for one-relator inverse monoids is at least as difficult as the word problem for one-relation semigroups.

Theorem

(Ivanov, Margolis, Meakin) If the word problem is decidable for every one relator inverse monoid of the form $Inv\langle X : w = 1\rangle$*, for* w *a reduced word, then the word problem for every one relation semigroup* $Sgp\langle X : u = v\rangle$ *(for* $u, v \in X^+$*) is decidable.*

We restrict to inverse monoids defined by a single defining relation corresponding to a cyclically reduced word w. Even in this case we are not able to solve the problem.

Problem 3: Is the word problem decidable for all one-relator inverse monoids of the form $Inv\langle X : w = 1\rangle$, for w a cyclically reduced word?

More on one-relator inverse monoids

Some positive information about inverse monoids of the form $M = Inv\langle X : w = 1\rangle$ when w is a *cyclically reduced* word in $(X \cup X^{-1})^*$ is provided by the following theorem, the proof of which makes heavy use of Schützenberger graphs, van Kampen diagrams, and an asphericity result for a class of two-relator groups due to Ivanov and Meakin.

Theorem

(Ivanov, Margolis, Meakin) Every inverse monoid of the form $M = Inv\langle X : w = 1\rangle$, where w is a cyclically reduced word, is E-unitary.

See the paper S.V. Ivanov, S.W. Margolis and J.C. Meakin, "On one-relator inverse monoids and one-relator groups", *J. Pure Appl. Algebra* 159 (2001), 83-111 for a proof of this result.

Problem 4: The Prefix Membership Problem

Since inverse monoids of the form $M = Inv\langle X : w = 1\rangle$, where w is a cyclically reduced word, are E-unitary, it follows that their Schützenberger graphs embed naturally in the Cayley graph of the corresponding one relator group $G = Gp\langle X : w = 1\rangle$. From this it follows that the word problem for M can be reduced to the corresponding prefix membership problem in G.

The prefix submonoid of G is the submonoid $P(w)$ of G generated by the prefixes of the word w. The corresponding prefix membership problem is the problem of deciding membership in the prefix submonoid $P(w)$ of G.

Problem 4: Is the prefix membership problem decidable for every cyclically reduced word w?

A positive solution to the prefix membership problem implies a positive solution to the word problem for the corresponding one-relator inverse monoid, but it is not known if the converse is true.

Unit cyclic conjugates

Let $M = Inv\langle X : w = 1\rangle$ where w is a cyclically reduced word. If w can be factored as $w \equiv uv$, then vu is a cyclic conjugate of w.

vu is called a *unit cyclic conjugate* of $w \equiv uv$ if $vu = 1$ in M. Then we can factor $w \equiv u_1 u_2 \dots u_k$ where $u_i \dots u_n u_1 \dots u_{i-1}$ are all the unit cyclic conjugates of w.

Proposition

Each u_i in the factorization above is in the group of units of M and the group of units is generated by the $u_i, i = 1, \dots k$. In particular, if w has no proper unit cyclic conjugates, then M has trivial group of units.

Question: Is the problem of finding all unit cyclic conjugates of w algorithmically equivalent to the word problem for M?

Problem 5: Amalgams of inverse semigroups

The structure of amalgamated free products of groups is well known. In particular the word problem for an amalgamated free product $G *_U H$ of groups is decidable if the ambient groups G and H have solvable word problems and the embeddings of U in G and H are sufficiently pleasant.

An early theorem about amalgams of inverse semigroups due to T.E. Hall guarantees that the category of inverse semigroups satisfies the strong amalgamation property (unlike the category of semigroups, that does not satisfy even the weak amalgamation property). On the other hand, the structure of an amalgamated free product $S *_U T$ of inverse semigroups is not well known.

Problem 5. Under what conditions on the inverse semigroups S and T and the embeddings of U in S and T is the word problem for $S *_U T$ in the category of inverse semigroups decidable?

A result of Cherubini, Meakin and Piochi shows that this is true if S and T are finite. (Not true for semigroups - Sapir)

Problem 6: Equations in Free Inverse Monoids

Celebrated theorems of Makanin show that the problem of deciding consistency of a finite system of equations in a free group or a free monoid is decidable. However, a beautiful theorem of Rozenblat shows that the corresponding problem for finite systems of equations in a free inverse monoid $FIM(A)$ is undecidable if $|A| > 1$.

On the other hand, Deis, Meakin and Senizergues made use of Rabin's tree theorem to prove that the problem of deciding whether a solution to a consistent system of equations in a free group $FG(A)$ may be extended to a solution to the same system of equations in the free inverse monoid $FIM(A)$ is decidable.

It is not known however, whether the consistency problem may be decidable for systems of certain specific equations, in particular, the following problem seems to be open.

Problem 6: Is the consistency problem for a single variable equation (i.e. an equation of the form $u(x, A) = v(x, A)$ involving constants from $A \cup A^{-1}$ and one variable x) in $FIM(A)$ decidable?