# How many Higher Commutator Operations can we Define on a Given Congruence Lattice of a Mal'cev Algebra?

Erhard Aichinger and Nebojša Mudrinski

Institute of Algebra, University Linz, Austria
Department of Mathematics and Informatics, University of Novi Sad, Serbia
erhard@algebra.uni-linz.ac.at, nmudrinski@dmi.uns.ac.rs

AAA83, Novi Sad, March 2012

# Motivation

PROBLEM: Describe all clones of operations on a finite set that contain all polynomial operations.

# Motivation

PROBLEM: Describe all clones of operations on a finite set that contain all polynomial operations.

HOPE: Every such a clone is determined by the set of all unary polynomials and all higher commutators.

# Higher Commutators

In general, introduced by A. Bulatov in 2001:

$$[\bullet, \ldots, \bullet]$$

as an $n$-ary operation, for each $n \geq 2$, on the lattice of congruences that satisfies the certain centralizing condition.

# Higher Commutators

In general, introduced by A. Bulatov in 2001:

$$[\bullet, \dots, \bullet]$$

as an $n$-ary operation, for each $n \geq 2$, on the lattice of congruences that satisfies the certain centralizing condition.

*Definition for expanded groups:*
**Theorem.** (E. Aichinger and $\sim$, published 2010) If $A_1, \dots, A_n \in \operatorname{Id} \mathbf{V}$, $\mathbf{V} = \langle V, +, F \rangle$ then $[A_1, \dots, A_n]$ is an ideal generated by the set

$$\{p(a_1, \dots, a_n) \,|\, a_i \in A_i, 1 \leq i \leq n, p \in \operatorname{Pol}_n \mathbf{V}$$

$$\text{such that } p(x_1, \dots, x_n) = 0 \text{ whenever } \exists i \text{ such that } x_i = 0\}.$$

# Properties of Higher Commutators in Mal'cev Algebras

**Theorem.** (E. Aichinger and ∼, published 2010)
Let **A** be a Mal'cev algebra, $n \geq 2$, $I \neq \emptyset$, and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \operatorname{Con} \mathbf{A}$

(HC1) $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge_{i=1}^n \alpha_i$

(HC2) $\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n \Rightarrow [\alpha_1, \ldots, \alpha_n] \leq [\beta_1, \ldots, \beta_n]$

(HC3) $[\alpha_1, \ldots, \alpha_n] \leq [\alpha_2, \ldots, \alpha_n]$

(HC4) $[\alpha_1, \ldots, \alpha_n] = [\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}]$,
for all permutations $\pi$ on $\{1, \ldots, n\}$

(HC7) $\bigvee_{i \in I}[\alpha_1, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_k] =$
$[\alpha_1, \ldots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_k]$,
for all $j \in \{1, \ldots, k\}$

(HC8) $[\alpha_1, \ldots, \alpha_j, [\alpha_{j+1}, \ldots, \alpha_k]] \leq [\alpha_1, \ldots, \alpha_k]$,
for all $j \in \{1, \ldots, k-2\}$.

# The Sequence of Higher Commutators

Bulatov obviously defined the sequence of higher commutators

$$[\bullet, \bullet], \ldots, [\bullet, \ldots, \bullet], \ldots$$

on the congruence lattice of an algebra - one operation for each arity not less then two.

# The Sequence of Higher Commutators

Bulatov obviously defined the sequence of higher commutators

$$[\bullet, \bullet], \ldots, [\bullet, \ldots, \bullet], \ldots$$

on the congruence lattice of an algebra - one operation for each arity not less then two.

This sequence satisfies the properties $(\mathrm{HC1}), (\mathrm{HC2}), (\mathrm{HC3}), (\mathrm{HC4}), (\mathrm{HC7}), (\mathrm{HC8})$.

# Sequences of Operations

Let $L$ be a lattice. We call the set $\langle f_i : L^i \to L \mid i \geq 2 \rangle$ a *sequence of operations* on the lattice $L$ if it satisfies the following properties:

(HC3) $f_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) \leq f_n(\alpha_2, \ldots, \alpha_{n+1})$

# Sequences of Operations

Let $L$ be a lattice. We call the set $\langle f_i : L^i \to L \mid i \geq 2 \rangle$ a *sequence of operations* on the lattice $L$ if it satisfies the following properties:

(HC3) $f_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) \leq f_n(\alpha_2, \ldots, \alpha_{n+1})$

(HC4) $f_n(\alpha_1, \ldots, \alpha_n) = f_n(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)})$ for all permutations $\pi$ on the set $\{1, \ldots, n\}$

## Sequences of Operations

Let $L$ be a lattice. We call the set $\langle f_i : L^i \to L \mid i \geq 2 \rangle$ a *sequence of operations* on the lattice $L$ if it satisfies the following properties:

(HC3) $f_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) \leq f_n(\alpha_2, \ldots, \alpha_{n+1})$

(HC4) $f_n(\alpha_1, \ldots, \alpha_n) = f_n(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)})$ for all permutations $\pi$ on the set $\{1, \ldots, n\}$

(HC8) $f_k(\alpha_1, \ldots, \alpha_{k-1}, f_{n-k+1}(\alpha_k, \ldots, \alpha_n)) \leq f_n(\alpha_1, \ldots, \alpha_n)$ where $k \in \{2, \ldots, n-1\}$ and $n \neq 2$,

for all $n \geq 2$ and for all $\alpha_1, \ldots, \alpha_n \in L$.

# Sequences With Additional Properties

We say that sequences of operations $\langle f_i : L^i \to L \mid i \geq 2 \rangle$ on a lattice $L$ satisfies

(HC1) if $f_n(\alpha_1, \ldots, \alpha_n) \leq \bigwedge_{i=1}^{n} \alpha_i$,

(HC2) if $\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n \Rightarrow f_n(\alpha_1, \ldots, \alpha_n) \leq f_n(\beta_1, \ldots, \beta_n)$,

(HC7) if $\bigvee_{i \in I} f_n(\alpha_1, \ldots, \alpha_{j-1}, \rho_i, \alpha_{j+1}, \ldots, \alpha_n) = f_n(\alpha_1, \ldots, \alpha_{j-1}, \bigvee_{i \in I} \rho_i, \alpha_{j+1}, \ldots, \alpha_n)$,

for all $n \geq 2$, $I \neq \emptyset$, $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in L$, $\{\rho_i \mid i \in I\} \subseteq L$ and $j \in \{1, \ldots, n\}$.

# Our Goal

Given: A congruence lattice $L$ of a finite Mal'cev algebra.

# Our Goal

Given: A congruence lattice $L$ of a finite Mal'cev algebra.

Question 1: How many sequences of operations that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$ can we define on $L$?

# Our Goal

Given: A congruence lattice $L$ of a finite Mal'cev algebra.

Question 1: How many sequences of operations that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$ can we define on $L$?

**Theorem.** (E. Aichinger, 2009) Let $L$ be a finite lattice. The number of sequences of functions $\langle f_1, f_2, \dots \rangle$ on $L$ that satisfy $(\mathrm{HC3})$ and $(\mathrm{HC4})$ is at most countable.

# Our Goal

Given: A congruence lattice $L$ of a finite Mal'cev algebra.

Question 1: How many sequences of operations that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$ can we define on $L$?

**Theorem.** (E. Aichinger, 2009) Let $L$ be a finite lattice. The number of sequences of functions $\langle f_1, f_2, \dots \rangle$ on $L$ that satisfy $(\mathrm{HC3})$ and $(\mathrm{HC4})$ is at most countable.

Answere to Question 1: There are at most countably many sequences of operations on $L$ that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

# Our Goal

Given: A congruence lattice $L$ of a finite Mal'cev algebra.

Question 1: How many sequences of operations that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$ can we define on $L$?

**Theorem.** (E. Aichinger, 2009) Let $L$ be a finite lattice. The number of sequences of functions $\langle f_1, f_2, \dots \rangle$ on $L$ that satisfy $(\mathrm{HC3})$ and $(\mathrm{HC4})$ is at most countable.

Answere to Question 1: There are at most countably many sequences of operations on $L$ that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

Question 2: Are there always infinitely many such sequences?

# The Splitting Property

**Definition.** Let $L$ be a lattice with the least element 0 and the largest element 1. We say that $L$ *splits* if

$$(\exists \delta, \epsilon \in L)\big(\delta < 1 \wedge \epsilon > 0 \wedge (\forall \alpha \in L)(\alpha \leq \delta \vee \alpha \geq \epsilon)\big).$$

Then, we call $(\delta, \epsilon)$ a *splitting pair*.

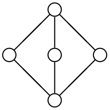
# The Splitting Property

**Definition.** Let $L$ be a lattice with the least element 0 and the largest element 1. We say that $L$ *splits* if

$$(\exists \delta, \epsilon \in L)\big(\delta < 1 \wedge \epsilon > 0 \wedge (\forall \alpha \in L)(\alpha \leq \delta \vee \alpha \geq \epsilon)\big).$$

Then, we call $(\delta, \epsilon)$ a *splitting pair*.

*Examples:*
The diamond does not split. The other three lattices on the picture split.



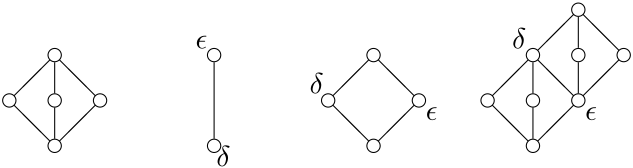
# The Splitting Property

**Definition.** Let $L$ be a lattice with the least element 0 and the largest element 1. We say that $L$ *splits* if

$$(\exists \delta, \epsilon \in L)\big(\delta < 1 \wedge \epsilon > 0 \wedge (\forall \alpha \in L)(\alpha \leq \delta \vee \alpha \geq \epsilon)\big).$$

Then, we call $(\delta, \epsilon)$ a *splitting pair*.

*Examples:*
The diamond does not split. The other three lattices on the picture split.



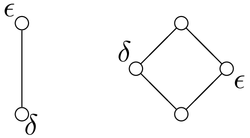
# The Strong Splitting Property

**Definition.** We say that $L$ *strongly splits* if it splits and

$$(\exists \delta, \epsilon \in L)\big(\delta < 1 \wedge \epsilon > 0 \wedge (\forall \alpha \in L)(\alpha \leq \delta \vee \alpha \geq \epsilon)$$

$$\wedge(\epsilon \leq \delta \text{ or } \epsilon \text{ is not an atom or } \delta \text{ is not a coatom})\big).$$

*Examples:* The two element lattice and $M_2$ do not split strongly, but the third lattice does. We denote it by $L_s$.



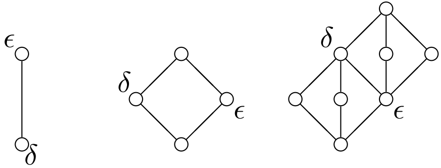
# The Strong Splitting Property

**Definition.** We say that $L$ *strongly splits* if it splits and

$$(\exists \delta, \epsilon \in L)\big(\delta < 1 \wedge \epsilon > 0 \wedge (\forall \alpha \in L)(\alpha \leq \delta \vee \alpha \geq \epsilon)$$

$$\wedge(\epsilon \leq \delta \text{ or } \epsilon \text{ is not an atom or } \delta \text{ is not a coatom})\big).$$

*Examples:* The two element lattice and $M_2$ do not split strongly, but the third lattice does. We denote it by $L_s$.



# The Diamond

**Proposition.** Let $f$ be a ternary function on the diamond that satisfies $(\mathrm{HC1})$ and $(\mathrm{HC7})$. Then, $f(1,1,1) = 0$.

# The Diamond

**Proposition.** Let $f$ be a ternary function on the diamond that satisfies $(\mathrm{HC1})$ and $(\mathrm{HC7})$. Then, $f(1,1,1) = 0$.

*Proof:* Let $\alpha, \beta, \gamma \notin \{0,1\}$. Using $(\mathrm{HC1})$ and $(\mathrm{HC7})$ we obtain

$$f(1,1,1) = f(\beta \vee \gamma, \alpha \vee \gamma, \alpha \vee \beta) = 0.$$

□

# The Diamond

**Proposition.** Let $f$ be a ternary function on the diamond that satisfies $(\mathrm{HC1})$ and $(\mathrm{HC7})$. Then, $f(1,1,1) = 0$.

*Proof:* Let $\alpha, \beta, \gamma \notin \{0,1\}$. Using $(\mathrm{HC1})$ and $(\mathrm{HC7})$ we obtain

$$f(1,1,1) = f(\beta \vee \gamma, \alpha \vee \gamma, \alpha \vee \beta) = 0.$$

□

**Proposition.** There are finitely many sequences of operations on the diamond that satisfy properties $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

# The Two Element Lattice

**Proposition.** Let $L$ be a lattice with the least element 0 and the largest element 1 and let $\theta$ be an atom of $L$. If $\langle f_k | k \geq 2 \rangle$ is a sequence of operations on $L$ that satisfies $(\mathrm{HC1})$ then $f_k(\theta, \ldots, \theta) = \theta$ for all $k \geq 2$ or $f_k(\theta, \ldots, \theta) = 0$ for all $k \geq 2$.

# The Two Element Lattice

**Proposition.** Let $L$ be a lattice with the least element 0 and the largest element 1 and let $\theta$ be an atom of $L$. If $\langle f_k | k \geq 2 \rangle$ is a sequence of operations on $L$ that satisfies $(\mathrm{HC1})$ then $f_k(\theta, \ldots, \theta) = \theta$ for all $k \geq 2$ or $f_k(\theta, \ldots, \theta) = 0$ for all $k \geq 2$.

*Proof:* Let us suppose that there exists an $n \in \mathbb{N}$ such that $f_n(\theta, \ldots, \theta) \neq 0$. Hence, $f_n(\theta, \ldots, \theta) = \theta$, because $\theta$ is an atom. We will prove that

$$f_k(\theta, \ldots, \theta) = \theta \text{ for all } k \geq 2. \quad \square$$

# The Two Element Lattice

**Proposition.** Let $L$ be a lattice with the least element 0 and the largest element 1 and let $\theta$ be an atom of $L$. If $\langle f_k | k \geq 2 \rangle$ is a sequence of operations on $L$ that satisfies $(\mathrm{HC1})$ then $f_k(\theta, \ldots, \theta) = \theta$ for all $k \geq 2$ or $f_k(\theta, \ldots, \theta) = 0$ for all $k \geq 2$.

*Proof:* Let us suppose that there exists an $n \in \mathbb{N}$ such that $f_n(\theta, \ldots, \theta) \neq 0$. Hence, $f_n(\theta, \ldots, \theta) = \theta$, because $\theta$ is an atom. We will prove that

$$f_k(\theta, \ldots, \theta) = \theta \text{ for all } k \geq 2. \quad \square$$

**Proposition.** There are only two sequences of operations that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$ on the two element lattice.

# How Do We Proceed?

**Proposition.** Let $\delta, \epsilon \in M_2 \backslash \{0, 1\}$. Then, for each $n \geq 2$ we have

$$f(\delta, \ldots, \delta) = g(\delta, \ldots, \delta) \wedge f(\epsilon, \ldots, \epsilon) = g(\epsilon, \ldots, \epsilon) \Rightarrow f = g,$$

for all $n$-ary functions $f$ and $g$ on $M_2$ that satisfy $(\mathrm{HC1})$ and $(\mathrm{HC7})$.

## How Do We Proceed?

**Proposition.** Let $\delta, \epsilon \in M_2 \backslash \{0, 1\}$. Then, for each $n \geq 2$ we have

$$f(\delta, \dots, \delta) = g(\delta, \dots, \delta) \wedge f(\epsilon, \dots, \epsilon) = g(\epsilon, \dots, \epsilon) \Rightarrow f = g,$$

for all $n$-ary functions $f$ and $g$ on $M_2$ that satisfy (HC1) and (HC7).

*Proof:* If $(x_1, \dots, x_n) \in \{\delta, \epsilon\}^n$ and $(x_1, \dots, x_n) \notin \{(\delta, \dots, \delta), (\epsilon, \dots, \epsilon)\}$ then, using (HC1) we have $f(x_1, \dots, x_n) = 0$.

# How Do We Proceed?

**Proposition.** Let $\delta, \epsilon \in M_2 \backslash \{0, 1\}$. Then, for each $n \geq 2$ we have

$$f(\delta, \ldots, \delta) = g(\delta, \ldots, \delta) \wedge f(\epsilon, \ldots, \epsilon) = g(\epsilon, \ldots, \epsilon) \Rightarrow f = g,$$

for all $n$-ary functions $f$ and $g$ on $M_2$ that satisfy $(\mathrm{HC1})$ and $(\mathrm{HC7})$.

*Proof:* If $(x_1, \ldots, x_n) \in \{\delta, \epsilon\}^n$ and $(x_1, \ldots, x_n) \notin \{(\delta, \ldots, \delta), (\epsilon, \ldots, \epsilon)\}$ then, using $(\mathrm{HC1})$ we have $f(x_1, \ldots, x_n) = 0$.

We know $\delta \vee \epsilon = 1$. Using $(\mathrm{HC7})$ we obtain the statement. □

# How Do We Proceed?

**Proposition.** Let $\delta, \epsilon \in M_2 \backslash \{0, 1\}$. Then, for each $n \geq 2$ we have

$$f(\delta, \ldots, \delta) = g(\delta, \ldots, \delta) \wedge f(\epsilon, \ldots, \epsilon) = g(\epsilon, \ldots, \epsilon) \Rightarrow f = g,$$

for all $n$-ary functions $f$ and $g$ on $M_2$ that satisfy $(\mathrm{HC1})$ and $(\mathrm{HC7})$.

*Proof:* If $(x_1, \ldots, x_n) \in \{\delta, \epsilon\}^n$ and $(x_1, \ldots, x_n) \notin \{(\delta, \ldots, \delta), (\epsilon, \ldots, \epsilon)\}$ then, using $(\mathrm{HC1})$ we have $f(x_1, \ldots, x_n) = 0$.

We know $\delta \vee \epsilon = 1$. Using $(\mathrm{HC7})$ we obtain the statement. □

**Proposition.** There are finitely many sequences of operations on the lattice $M_2$ that satisfy properties $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

# The Example of a Strongly Splitting Lattice

**Proposition.** Let $n \geq 2$ and $(\delta, \epsilon)$ the splitting pair of the lattice $L_s$. If we define $f_n : L_s^n \longrightarrow L_s$ such that

$$f_n(\alpha_1, \ldots, \alpha_n) := \begin{cases} 0 & , (\exists i) \alpha_i \leq \delta \\ \epsilon & , \text{ otherwise,} \end{cases}$$

then $f_n$ satisfies $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

# The Example of a Strongly Splitting Lattice

**Proposition.** Let $n \geq 2$ and $(\delta, \epsilon)$ the splitting pair of the lattice $L_s$. If we define $f_n : L_s^n \to L_s$ such that

$$f_n(\alpha_1, \ldots, \alpha_n) := \begin{cases} 0 & , (\exists i)\alpha_i \leq \delta \\ \epsilon & , \text{ otherwise,} \end{cases}$$

then $f_n$ satisfies $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

**Proposition.** The following sequences

$$\langle f_2, 0, 0, \ldots \rangle, \langle f_2, f_3, 0, \ldots \rangle, \ldots$$

are infinitely many sequences of operations on the lattice $L_s$ that satisfy $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$.

# The Answer

**Theorem.** Let **A** be a Mal'cev algebra with a finite congruence lattice $L$. There are infinitely many sequences of operations on the lattice $L$ that satisfy properties $(\mathrm{HC1}), (\mathrm{HC2})$ and $(\mathrm{HC7})$ if and only if $L$ splits strongly.