Universal homogeneous constraint structures and the hom-equivalence classes of weakly oligomorphic structures

Christian Pech
Maja Pech

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Weakly oligomorphic structures

Definition

A countable relational structure **A** is called weakly oligomorphic if $\mathrm{End}(\mathbf{A})$ is oligomorphic. I.e., $\mathrm{End}(\mathbf{A})$ has of every arity only finitely many invariant relations on A.

Examples for weakly oligomorphic structures

- finite structures,
- ω-categorical structures,
- retracts of weakly oligomorphic structures,
- reducts of homomorphism homogeneous structures over a finite signature

Motivation

Define $\mathrm{CSP}(\mathbf{A}) := \{\mathbf{B} \mid \mathbf{B} \text{ finite}, \mathbf{B} \to \mathbf{A}\}$

Theorem

If $\mathbf{B}$ *is weakly oligomorphic and* $\mathbf{A}$ *is a countable structure, then the following are equivalent:*

1. $\mathbf{A} \to \mathbf{B}$,
2. $\mathrm{Th}^{\exists_1^+}(\mathbf{A}) \subseteq \mathrm{Th}^{\exists_1^+}(\mathbf{B})$,
3. $\mathrm{Age}(\mathbf{A}) \to \mathrm{Age}(\mathbf{B})$,
4. $\mathrm{CSP}(\mathbf{A}) \subseteq \mathrm{CSP}(\mathbf{B})$.

Theorem (Mašulović, MP ’11)

If $\mathbf{A}$ *is weakly oligomorphic and* $\mathbf{B}$ *is countable and* $\mathbf{B} \models \mathrm{Th}(\mathbf{A})$, *then* $\mathbf{B}$ *is weakly oligomorphic.*

Corollary

Let T *be the first order theory of a weakly oligomorphic structure. Then all countable models of* T *are homomorphism-equivalent.*

Hom-equivalence classes

Definition

Let $\mathbf{A}$ be a countable relational structure. Then the hom-equivalence class $\mathcal{E}(\mathbf{A})$ of $\mathbf{A}$ is the class of all countable structures $\mathbf{B}$ such that $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{A}$.

We equip $\mathcal{E}(\mathbf{A})$ with a quasiorder:

For $\mathbf{B}, \mathbf{C} \in \mathcal{E}(\mathbf{A})$ we write $\mathbf{B} \hookrightarrow \mathbf{C}$ whenever there exists an embedding from $\mathbf{B}$ into $\mathbf{C}$.

We study the structure of $(\mathcal{E}(\mathbf{A}), \hookrightarrow)$,

where $\mathbf{A}$ is a weakly oligomorphic structure.
Our first steps are to find (nice) smallest and greatest elements in $\mathcal{E}(\mathbf{A})$.

Smallest elements

Theorem

Every weakly oligomorphic relational structure $\mathbf{T}$ *is homomorphism-equivalent to a finite or* $\aleph_0$*-categorical substructure* $\mathbf{C}$.

Theorem (Bodirsky '07)

Every $\aleph_0$*-categorical relational structure* $\mathbf{T}$ *is homomorphism-equivalent to a model-complete core* $\mathbf{C}$, *which is unique up to isomorphism, and* ω*-categorical or finite. . . .*

Corollary

For a weakly oligomorphic structure $\mathbf{A}$ *the class* $\mathcal{E}(\mathbf{A})$ *has (up to isomorphism) a unique model-complete smallest element.*

Greatest elements

Theorem

Let R be a countable relational signature, and let $\mathbf{T}$ be a countable R-structure. Then $\mathcal{E}(\mathbf{T})$ has a largest element. Moreover, if R is finite and $\mathbf{T}$ is weakly oligomorphic, then $\mathcal{E}(\mathbf{T})$ has an ω-categorical element.

Theorem (Saracino '73)

Let T be an $\aleph_0$-categorical theory with no finite models. Then T has a model-companion T'. Moreover, T' is $\aleph_0$-categorical, too.

Corollary

If $\mathbf{A}$ is a weakly oligomorphic structure over a finite signature, then $\mathcal{E}(\mathbf{A})$ has (up to isomorphism) a unique model-complete, ω-categorical largest element.

Observation

The age of a largest element in $\mathcal{E}(\mathbf{A})$ is at most CSP($\mathbf{A}$).

Strict Fraïssé-classes

If $\mathcal{C}$ is an age, then $\overline{\mathcal{C}} := \{\mathbf{A} \mid \mathbf{A} \text{ countable}, \operatorname{Age}(\mathbf{A}) \subseteq \mathcal{C}\}$.

Definition (Dolinka)

A Fraïssé-class $\mathcal{C}$ of relational structures is called strict Fraïssé-class if every pair of morphisms in $(\mathcal{C}, \hookrightarrow)$ with the same domain has a finite pushout in $(\overline{\mathcal{C}}, \to)$.

Observation

Note that these pushouts will always be amalgams. Thus the strict amalgamation property postulates canonical amalgams.

Examples for strict Fraïssé-classes

- free amalgamation classes,
- the class of finite partial orders.

Definition

A sub-Fraïssé-class $\mathcal{C}$ of a strict Fraïssé-class $\mathcal{U}$ is called free in $\mathcal{U}$ if it is closed with respect to canonical amalgams.

Universal structures

Theorem

Let $\mathcal{U}$ be a strict Fraïssé-class of relational structures, and let $\mathcal{C}$ be a Fraïssé-class that is free in $\mathcal{U}$. Let $\mathbf{T} \in \overline{\mathcal{U}}$. Then

1. $\overline{\mathcal{C}} \cap \overline{\mathrm{CSP}(\mathbf{T})}$ *has a universal element* $\mathbf{U}_{\mathcal{C},\mathbf{T}}$*,*
2. *if the Fraïssé-limit of $\mathcal{C}$ and $\mathbf{T}$ each have an oligomorphic automorphism group (i.e. each is finite or ω-categorical), then $\overline{\mathcal{C}} \cap \overline{\mathrm{CSP}(\mathbf{T})}$ has a universal element $\mathbf{U}_{\mathcal{C},\mathbf{T}}$ that is finite or ω-categorical.*

If $\mathbf{T} \in \overline{\mathcal{C}}$, then $\mathbf{U}_{\mathcal{C},\mathbf{T}}$ can be chosen as a co-retract of $\mathbf{T}$.

Special case

R is a countable relational signature, $\mathbf{T}$ an *R*-structure, and $\mathcal{U} = \mathcal{C}$ is the class of all finite *R*-structures.

T-colored structures

Given

- a strict Fraïssé-class $\mathcal{U}$,
- a Fraïssé-class $\mathcal{C}$, that is free in $\mathcal{U}$, and
- $\mathbf{T} \in \overline{\mathcal{U}}$.

Definition

A **T**-colored structure in $\overline{\mathcal{C}}$ is a pair $(\mathbf{A}, a)$ such that $\mathbf{A} \in \overline{\mathcal{C}}$ and $a : \mathbf{A} \to \mathbf{T}$ is a homomorphism. The class of all such structures is denoted by $\mathrm{Col}_{\mathcal{C}}(\mathbf{T})$.

Note

A countable structure $\mathbf{A}$ is in $\overline{\mathcal{C}} \cap \overline{\mathrm{CSP}(\mathbf{T})}$ if and only if there exists $f : \mathbf{A} \to \mathbf{T}$ such that $(\mathbf{A}, a)$ is a **T**-colored structure in $\overline{\mathcal{C}}$.

Morphisms for $\mathbf{T}$-colored structures

Strong homomorphisms

$f : (\mathbf{A}, a) \to (\mathbf{B}, b)$ is called a strong homomorphism if $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism and $b \circ f = a$. Analogously strong embeddings and strong automorphisms are defined. $\mathrm{sAut}(\mathbf{A}, a)$ denotes the group of strong automorphisms.

Weak homomorphisms

A weak homomorphism from $(\mathbf{A}, a)$ to $(\mathbf{B}, b)$ is a pair (f, g) such that $f : \mathbf{A} \to \mathbf{B}$, $g \in \mathrm{Aut}(\mathbf{T})$, $b \circ f = g \circ a$. If f is an embedding (an automorphism), then (f, g) is called a weak embedding (a weak automorphism). Composition is component-wise.
$\mathrm{wAut}(\mathbf{A}, a)$ denotes the group of weak automorphisms.
$\mathrm{cAut}(\mathbf{A}, a) := \{f \in \mathrm{Aut}(\mathbf{A}) \mid \exists g \in \mathrm{Aut}(\mathbf{T}) : (f, g) \in \mathrm{wAut}(\mathbf{A}, a)\}$.

Remark

- *We have* $f : (\mathbf{A}, a) \to (\mathbf{B}, b)$ *iff* $(f, 1_{\mathbf{T}}) : (\mathbf{A}, a) \to (\mathbf{B}, b)$.
- *If* a *is surjective, then* $\mathrm{cAut}(\mathbf{A}, a) \cong \mathrm{wAut}(\mathbf{A}, a)$.

Universal homogeneous $\mathbf{T}$-colored structures

Theorem

There exists $(\mathbf{U}, u) \in \mathrm{Col}_{\mathcal{C}}(\mathbf{T})$ *such that*

1. *for every* $(\mathbf{A}, a) \in \mathrm{Col}_{\mathcal{C}}(\mathbf{T})$ *there exists an embedding* $\iota : (\mathbf{A}, a) \hookrightarrow (\mathbf{U}, u)$ *(universality),*
2. *for every finite* $(\mathbf{A}, a) \in \mathrm{Col}_{\mathcal{C}}(\mathbf{T})$*, and for all* $\iota_1, \iota_2 : (\mathbf{A}, a) \hookrightarrow (\mathbf{U}, u)$ *there exists* $f \in \mathrm{sAut}(\mathbf{U}, u)$ *such that* $f \circ \iota_1 = \iota_2$ *(homogeneity).*

Moreover, all countable universal homogeneous $\mathbf{T}$*-colored structures are mutually isomorphic.*

Remark

- *If* $\text{F-Lim}(\mathcal{C})$ *is finite or* ω*-categorical, and if* $\mathbf{T}$ *is finite, then* $\mathrm{sAut}(\mathbf{U}, u)$ *is oligomorphic.*
- *If* $\mathbf{T} \in \overline{\mathcal{C}}$*, then* $\mathbf{T}$ *is a retract of* $\mathbf{U}$*.*

w-homogeneity

Definition

$(\mathbf{U}, u) \in \mathrm{Col}_{\mathcal{C}}(\mathbf{T})$ is called w-homogeneous if for every finite $(\mathbf{A}, a) \in \mathrm{Col}_{\mathcal{C}}(\mathbf{T})$, and for $(f_1, g_2), (f_2, g_2) : (\mathbf{A}, a) \hookrightarrow (\mathbf{U}, u)$ there exists $(f, g) \in \mathrm{wAut}(\mathbf{U}, u)$ such that $(f, g) \circ (f_1, g_1) = (f_2, g_2)$.

Proposition

Let $(\mathbf{U}, u) \in \mathrm{Col}_{\mathcal{C}}(\mathbf{T})$ *be universal and homogeneous. Then* $(\mathbf{U}, u)$ *is w-homogeneous, too.*

Remark

- *If* $\mathrm{F\text{-}Lim}(\mathcal{C})$ *is finite or* ω*-categorical, and if* $\mathbf{T}$ *is finite or* ω*-categorical, too, then* $\mathrm{cAut}(\mathbf{U}, u)$ *is oligomorphic.*

Universal homogeneous objects in categories

Definition

We call a category $\mathfrak{C}$ a λ-amalgamation category if

1. all morphisms of $\mathfrak{C}$ are monomorphisms,
2. $\mathfrak{C}$ is λ-algebroidal,
3. $\mathfrak{C}_{<\lambda}$ has the joint embedding property,
4. $\mathfrak{C}_{<\lambda}$ has the amalgamation property.

Theorem (Droste, Göbel '92)

Let λ be a regular cardinal, and let $\mathfrak{C}$ be a λ-algebroidal category in which all morphisms are monomorphisms. Then there exists a $\mathfrak{C}$-universal, $\mathfrak{C}_{<\lambda}$-homogeneous object in $\mathfrak{C}$ if and only if $\mathfrak{C}$ is a λ-amalgamation category. Moreover, any two $\mathfrak{C}$-universal, $\mathfrak{C}_{<\lambda}$-homogeneous objects in $\mathfrak{C}$ are isomorphic.

Amalgamation pairs

Definition

A pair of categories $(\mathfrak{A}, \widehat{\mathfrak{A}})$ is called a λ-amalgamation pair if

1. $\mathfrak{A} \leq \widehat{\mathfrak{A}}$ is isomorphism closed,
2. all morphisms of $\mathfrak{A}$ are monomorphisms,
3. $\mathfrak{A}$ is λ-algebroidal,
4. $\mathfrak{A}_{<\lambda}$ has the free joint embedding property in $\widehat{\mathfrak{A}}$, and
5. $\mathfrak{A}_{<\lambda}$ has the free amalgamation property in $\widehat{\mathfrak{A}}$.

Remark

λ-amalgamation pairs are a category-theoretic version of the idea of free amalgamation classes and of strict amalgamation classes

Theorem

Let $(\widehat{\mathfrak{A}}, \mathfrak{A})$ be a λ-amalgamation pair, $\mathfrak{B}$ be a λ-amalgamation category, and let $\mathfrak{C}$ be a category. Let $\widehat{F} : \widehat{\mathfrak{A}} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$ and let F be the restriction of $\widehat{F}$ to $\mathfrak{A}$. Further suppose that

1. *$\widehat{F}$ preserves weak coproducts and weak pushouts in $\mathfrak{A}_{<\lambda}$,*
2. *F and G are λ-continuous,*
3. *F preserves λ-smallness,*
4. *G preserves monomorphisms,*
5. *for every $A \in \mathfrak{A}_{<\lambda}$ and for every $B \in \mathfrak{B}_{<\lambda}$ there are at most λ morphisms in $\mathfrak{C}(FA \to GB)$.*

Then $(F \downarrow G)$ has a $(F \downarrow G)$-universal, $(F \downarrow G)_{<\lambda}$-homogeneous object. Moreover, up to isomorphism there is just one such object in $(F \downarrow G)$.

Definition

A Fraïssé-class $\mathcal{C}$ has the Hrushovski property if for every $\mathbf{A} \in \mathcal{C}$ there exists a $\mathbf{B} \in \mathcal{C}$ such that $\mathbf{A} \leq \mathbf{B}$ and such that every isomorphism between substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

Definition

Let $G \leq S_\omega$. Then G is said to have the small index property if every subgroup of index less than $2^{\aleph_0}$ contains the stabilizer of a finite tuple (i.e. subgroups of small index are open in the topology of pointwise convergence on G).

Remark

- *The Hrushovski-property of a free amalgamation class $\mathcal{C}$ implies the small index property of the automorphism group of* F-Lim($\mathcal{C}$).
- *The Hrushovski-property can straight-forwardly be defined for Fraïssé-classes of finite constraint structures.*

Link-structures

A finite R-structure $\mathbf{A}$ is called a link-structure, if either $|A| = 1$ or there exist $a_1, \ldots, a_n \in A$ such that $A = \{a_1, \ldots, a_n\}$ and for some $\varrho \in R^{(n)}$ we have $(a_1, \ldots, a_n) \in \varrho_{\mathbf{A}}$.

Link-type

If $\mathcal{L}$ is a set of link-structures, then we say that a structure $\mathbf{A}$ has link type $\mathcal{L}$ if every substructure of $\mathbf{A}$ that is a link structure, is isomorphic to some structure from $\mathcal{L}$.

Free monotone amalgamation classes

A free amalgamation class is called monotone if it is a CSP, too.

Definition

Let $\mathcal{C}$ be a free monotone amalgamation class, $\mathcal{L}$ be a set of link-structures. By $\mathcal{C}_{\mathcal{L}}$ we denote the class of all structures from $\mathcal{C}$ whose link-type is $\mathcal{L}$.

Remark

$\mathcal{C}_{\mathcal{L}}$ is a free amalgamation class, too.

Definition

A finite structure is called sparse if it has only finitely many non-empty basic relations. A relational structure is called sparse if all finite substructures are sparse.

Theorem

Let R be any relational signature, let $\mathcal{C}$ be a free, monotone amalgamation class, and let $\mathcal{L}$ be a countable set of sparse link-structures. Let $\mathbf{T}$ be any countable R-structure. Then $\mathrm{Col}_{\mathcal{C}_{\mathcal{L}}}(\mathbf{T})$ *has the Hrushovski property. If $(\mathbf{U}, u)$ is a universal homogeneous $\mathbf{T}$-colored structure in $\overline{\mathcal{C}_{\mathcal{L}}}$, then* $\mathrm{sAut}(\mathbf{U}, u)$ *has the small index property.*

Remark

- *The proof uses an adapted version of a criterion for the (SIP) due to Herwig (which in turn generalizes Hrushovski's ideas from graphs to relational structures).*
- *If* $\mathrm{sAut}(\mathbf{U}, u)$ *is oligomorphic, then it has uncountable cofinality and the Bergman-property. (Kechris, Rosendal)*