Galois connections between group actions and functions – some results and problems

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Outline

Definition

The Galois connection induced by a binary relation

$$R \subseteq G \times M$$

is given by the pair of mappings

$$\varphi : \mathfrak{P}(G) \to \mathfrak{P}(M) : X \mapsto X^R := \{m \in M \mid \forall g \in X : gRm\}$$
$$\psi : \mathfrak{P}(M) \to \mathfrak{P}(G) : Y \mapsto Y^R := \{g \in G \mid \forall m \in Y : gRm\}$$

Galois closures $X = (X^R)^R$, $Y = (Y^R)^R$

A Galois connection (φ, ψ) is characterizable by the property

$$\forall X \subseteq G, Y \subseteq M : Y \subseteq \varphi(X) \iff \psi(Y) \supseteq X$$

In Formal Concept Analysis (FCA)(*Ganter/Wille*):
G: objects (Gegenstände), M: attributes (Merkmale),
gRm: object g has attribute m

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Examples

$R = \models$: $\mathcal{A} \models s \approx t$ (algebra satisfies term equation)
Galois closures:

$(\mathcal{K}^{\models})^{\models} = \operatorname{Mod}\operatorname{Id}\mathcal{K}$ equational classes = varieties

$(\Sigma^{\models})^{\models} = \operatorname{Id}\operatorname{Mod}\Sigma$ equational theories

$R = \triangleright$: $f \triangleright \varrho$ (function preserves relation)
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$(F^{\triangleright})^{\triangleright} = \operatorname{Pol}\operatorname{Inv} F$ clones

$(Q^{\triangleright})^{\triangleright} = \operatorname{Inv}\operatorname{Pol} Q$ relational clones

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$\Gamma = (\Gamma, \cdot, \varepsilon)$ group (with identity element ε)

(A, Γ) group action (Γ acts on a set A): mapping

$$A \times \Gamma \to A : (a, \sigma) \mapsto a^{\sigma}$$

such that

$$x^{\varepsilon} = x$$
$$(x^{\sigma})^{\tau} = x^{\sigma\tau}$$

for all $x \in A$ and $\sigma, \tau \in \Gamma$.

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Examples of group actions

- Permutation groups $G \leq \Gamma := \mathrm{Sym}(A)$ acting on set A: natural action (A, G) on A: $a^\sigma := \sigma(a)$ for $a \in A$, $\sigma \in G$.

- Permutation groups $G \leq \Gamma := \mathrm{Sym}(\underline{n})$ acting on $A := \mathbf{2}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbf{2}\}$ (where $\mathbf{2} := \{0, 1\}$): action: $(x_1, \ldots, x_n)^\sigma := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

- Permutation groups $G \leq \Gamma := \mathrm{Sym}(\underline{n})$ acting on $A := \mathfrak{P}(\underline{n})$: action: $B^\sigma := \{\sigma(b) \mid b \in B\}$ for $B \subseteq \underline{n} := \{1, \ldots, n\}$.

- $\Gamma := \mathrm{GL}_n(2)$ (general linear group) acting on $A := \mathbf{2}^n$: action of a regular $(n \times n)$-matrix $M \in \mathrm{GL}_n(2)$ (over 2-element field GF(2)) on $\vec{x} = (x_1, \ldots, x_n)^\top$ (considered as column vector) by matrix multiplication: $x^M := M\vec{x}$, (all computations in GF(2)).

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The Galois connection induced by $\vdash$

(A, Γ) group action, K arbitrary set (e.g. $K = \mathbf{2}$)
$\vdash$ relation between group elements $\sigma \in \Gamma$ and functions $f : A \to K$

Definition

$$\sigma \vdash f :\iff \forall x \in A : f(x^\sigma) = f(x).$$

$$\begin{array}{ccc} A & \xrightarrow[x \mapsto x^\sigma]{\text{action}} & A \\ \| & & \downarrow f \\ A & \xrightarrow{f} & K \end{array}$$

Then $f \in K^A$ is called an invariant for $\sigma \in \Gamma$ and σ is called a symmetry of f.

Clearly, $\sigma \vdash f$ if and only if $\sigma^{-1} \vdash f$.

Corresponding Galois connection (let $F \subseteq K^A$ and $G \subseteq \Gamma$)

$$F^\vdash := \{\sigma \in \Gamma \mid \forall f \in F : \sigma \vdash f\}, \; G^\vdash := \{f \in K^A \mid \forall \sigma \in G : \sigma \vdash f\},$$

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The problem and preliminary notions

easy to check:
Galois closures $\overline{G} = (G^{\vdash})^{\vdash}$ are always subgroups of Γ.

Problem

Given a group action (A, Γ), characterize the Galois closed subgroups $G = \overline{G}$.

some necessary notions and notation:
For a subgroup $G \leq \Gamma$ let

$$\mathrm{Orb}_A\, G := \{a^G \mid a \in A\} \text{ (where } a^G := \{a^\sigma \mid \sigma \in G\})$$

(set of all orbits of G (under the group action)).
For $a \in A$ and $B \subseteq A$ let

$$\Gamma_a := \{\sigma \in \Gamma \mid a^\sigma = a\} \quad \text{(stabilizer of } a).$$

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Characterizing $\sigma \in G$

Lemma

The following conditions are equivalent (for $G \leq \Gamma$, $f \in K^A$):

(i) $f \in G^{\vdash}$,

(ii) f *is constant on each* $B \in \operatorname{Orb} G$,

Proof.

Directly follows from

$b, b' \in B \in \operatorname{Orb}(G) \iff \exists \sigma \in G : b' = b^{\sigma}$. □

Characterization Theorem

Theorem

Let (A, Γ) be a group action and $G \leq \Gamma$. Then we have:

$$\overline{G} = \bigcap_{B \in \mathrm{Orb}(G)} \Gamma_B, \tag{*}$$

$$\overline{G} = \bigcap_{a \in A} \Gamma_a \cdot G. \tag{**}$$

Moreover, the Galois closure $\overline{G}$ is the largest subgroup among all subgroups of Γ with the same orbits (on A) as G.

Remark: For the action $(A, \Gamma) = (\mathbf{2}^n, \mathrm{Sym}(\underline{n}))$, (**) was formulated and proved by *E. Horvath, K. Kearnes*.

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Proof of (*)

(*) $$\overline{G} = \bigcap_{B \in \mathrm{Orb}(G)} \Gamma_B = \bigcap_{B \in \mathrm{Orb}(G)} \{\sigma \in \Gamma \mid B^\sigma = B\}.$$

Proof.

"$\supseteq$": Let $\sigma \in \Gamma$ satisfy $B^\sigma = B$ for each orbit B. Every $f \in G^\vdash$ is constant on each orbit, and, for each $b \in A$, b, b^σ belong to the same orbit by assumption, therefore we have $f(b) = f(b^\sigma)$. Thus $\sigma \vdash f$, consequently $\sigma \in (G^\vdash)^\vdash = \overline{G}$.

"$\subseteq$": Let $\sigma \in \overline{G}$ and $B \in \mathrm{Orb}(G)$. We define $f_B : A \to K$ by

$$f_B(a) := \begin{cases} 1 & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f_B \in G^\vdash$ because it is constant on each orbit. Consequently $\sigma \vdash f_B$, in particular $f(b^\sigma) = f(b) = 1$ for each $b \in B$, i.e. $b^\sigma \in B$ by definition of f_B. Thus $B^\sigma = B$.

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"$\supseteq$": Let $\sigma \in \Gamma_a \cdot G$ for all $a \in A$. Then, for each $a \in A$, there exists $\tau_a \in \Gamma_a$ and $\pi_a \in G$ such that $\sigma = \tau_a \pi_a$. Let $f \in G^{\vdash}$, then $\pi_a \vdash f$, thus $f(a^{\pi_a}) = f(a)$. Because $a^{\tau_a} = a$ we get $f(a^\sigma) = f(a^{\tau_a \pi_a}) = f(a^{\pi_a}) = f(a)$, showing that $\sigma \vdash f$, consequently $\sigma \in (G^{\vdash})^{\vdash} = \overline{G}$.

"$\subseteq$": Let $\sigma \in \overline{G}$, $a \in A$ and $B = a^G \in \mathrm{Orb}(G)$. By (*) we have $a^\sigma \in B = a^G$. From the last equation we see that there exists a $\pi \in G$ with $a^\sigma = a^\pi$. Hence $a^{\sigma\pi^{-1}} = a$, and we have $\sigma = (\sigma\pi^{-1})\pi \in \Gamma_a \cdot G$. □

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TECHNISCHE UNIVERSITÄT DRESDEN

Characterization for the natural action $(A, \mathrm{Sym}(A))$

Proposition

Let $\Gamma = \mathrm{Sym}(A)$ *be the full symmetric group with its natural action on* A*. The Galois closed subgroups* G *of* S_A *are exactly those of the form*

$$G = \mathrm{Sym}_A(B_1) \cdot \ldots \cdot \mathrm{Sym}_A(B_r) \cong \mathrm{Sym}(B_1) \times \ldots \times \mathrm{Sym}(B_r),$$

where $\{B_1, \ldots, B_r\}$ *is a partition of* A*.*
Then $\mathrm{Orb}(G) = \{B_1, \ldots, B_r\}$*.*

For $B \subseteq A$, here $\mathrm{Sym}_A(B)$ denotes the image of the natural embedding $\sigma \mapsto \hat{\sigma}$ of $\mathrm{Sym}(B)$ into $\mathrm{Sym}(A)$ where, for $a \in A$,

$$a^{\hat{\sigma}} := \begin{cases} a^{\sigma} & \text{if } a \in B, \\ a & \text{otherwise.} \end{cases}$$

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Generalization to faithful actions

Proposition

Let Γ be a faithful action on A. The Galois closed subgroups G of Γ are exactly those of the form

$$\hat{G} = \mathrm{Sym}_A(B_1) \cdot \ldots \cdot \mathrm{Sym}_A(B_r) \cap \hat{\Gamma},$$

where $\{B_1, \ldots, B_r\}$ is a partition of A.

Here $\hat{\Gamma}$ (and $\hat{G}$) denotes the natural permutation representation of the group action:

$$\hat{\Gamma} := \{\hat{\sigma} \mid \sigma \in \Gamma\} \text{ where } \hat{\sigma} : A \to A : x \mapsto x^{\sigma}.$$

(faithful action $\implies \Gamma \cong \hat{\Gamma}$)

Outline

Some Problems

- Find the Galois closed subgroups for concrete actions (A, Γ) .
  Characterize the Galois closed groups of the form $G = \{f\}^{\vdash}$ for a single function $f : A \to K$ (e.g. with $K = \mathbf{2}$).
  For finite actions: every closed G is of this form if the size of K is chosen large enough (e.g. $K = \mathbf{2}^A$).
- The other side of the Galois connection:
  Find and characterize the Galois closed sets $\overline{F} = F \subseteq K^A$ of functions $f : A \to K$.
- Which generalizations make sense:
  groups $\to$ semigroups ?
  functions $\to$ other objects ?

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some further results for the action $(\mathbf{2}^n, \mathrm{Sym}(\underline{n}))$ by

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Thank you for your ATTENTION!