Generating Direct Powers

Nik Ruškuc

nik@mcs.st-and.ac.uk

School of Mathematics and Statistics, University of St Andrews

Novi Sad, 16 March 2012

University of St Andrews

Algebraic structures

- Classical: groups, rings, modules, algebras, Lie algebras.
- Semigroups.
- Modern: lattices, boolean algebras, loops, tournaments, relational algebras, universal algebras,...

The **d**-sequence

For an algebraic structure A:

- $d(A)$ = the smallest number of generators for A.
- $A^n = \{(a_1, \ldots, a_n) \, : \, a_i \in A\}$.
- $\mathbf{d}(A) = (d(A), d(A^2), d(A^3), \ldots)$.

Some basic properties:

- $\mathbf{d}(A)$ is non-decreasing.
- $\mathbf{d}(A)$ is bounded above by $|A|^n$.
- If A has an 'identity element' then $d(A^n) \leq nd(A)$.

General problem

Relate algebraic properties of A with numerical properties (e.g. the rate of growth) of its $\mathbf{d}$ sequence.

Groups

Jim Wiegold and collaborators, 1974–89.

- $\mathbf{d}(G)$ is linear if G is non-perfect ($G' \neq G$);
- $\mathbf{d}(G)$ is logarithmic if G is finite and perfect;
- $\mathbf{d}(G)$ is bounded above by a logarithmic function if G is infinite and perfect;
- $\mathbf{d}(G)$ is eventually constant if G is infinite simple.

Open Problem

Can $\mathbf{d}(G)$ be strictly between constant and logarithmic?

Open Problem

Does there exist an infinite simple group G such that $d(G^n) = d(G) + 1$ for some n?

Classical structures

Martyn Quick, NR.

Theorem

The $\mathbf{d}$-sequence of a finite non-trivial classical structure grows either logarithmically or linearly. Those with logarithmic growth are: perfect groups, rings with 1, algebras with 1, and perfect Lie algebras.

Theorem

The $\mathbf{d}$-sequence of an infinite classical structure grows either linearly or sub-logarithmically. Simple structures have eventually constant $\mathbf{d}$-sequences.

Congruence permutable varieties

Arthur Geddes; Peter Mayr.

Theorem

The **d***-sequence of a finite non-trivial structure belonging to a congruence permutable variety is either logarithmic or linear.*

Theorem

The **d***-sequence of an infinite structure belonging to a congruence permutable variety grows either linearly or sub-logarithmically. Simple structures have eventually constant* **d***-sequences.*

Some other structures

- Lattices: sub-logarithmic. (Geddes)
- Finite tournaments: linear or logarithmic. (Geddes)
- 2-element algebras: logarithmic, linear or exponential. (St Andrews summer students)
- There exist 3-element algebras with polynomial growth of arbitrary degree. (Geddes; Kearnes, Szendrei?)

Representation Theorem

Theorem (Geddes)

For every non-decreasing sequence $\mathbf{s}$ *there exists an algebraic structure* A *with* $\mathbf{d}(A) = \mathbf{s}$.

Open Problem

Characterise the $\mathbf{d}$-sequences of finite algebraic structures.

Sequences in algebra

- Grätzer et al.: p_n-sequences, free spectra.
- Berman et al. (2009): three sequences **s**, **g**, **i** to do with subuniverses of $\mathbf{A^n}$ and their generating sets.

Theorem (Kearnes, Szendrei?)

The **d***-sequence of a finite algebraic structure with few subpowers is either logarithmic or linear.*

Another direction: quantified constraint satisfaction (Chen).

Intermezzo: an elementary question

Very important. Would you ask an understanding and indulgent maths colleague how many digits there would be in the result of multiplying $1 \times 2 \times 3 \times 4$ *etc. up to* 1000 *(*1000 *being the last multiplier and the product of all numbers from* 1 *to* 999 *being the last multiplicand). If there is any way of obtaining the exact result (but here I have the feeling that I am raving) without too much drudgery, by using for example logarithms or a calculator, I'm all for it. But in any event how many figures overall. I'll be satisfied with that.* (S. Beckett to M. Peron, 1952)

Finite semigroups

Example

If S is a left zero semigroup ($xy = x$) then

$$\mathbf{d}(S) = (|S|, |S|^2, |S|^3, \dots).$$

Theorem (Wiegold 1987)

For a finite (non-group) semigroup S we have:

- $\mathbf{d}(S)$ *is linear if S is a monoid;*
- *otherwise* $\mathbf{d}(S)$ *is exponential.*

Infinite semigroups: how bad can they get?

Example

$\mathbf{d}(\mathbb{N}) = (1, \infty, \infty, \dots)$.

Theorem (EF Robertson, NR, J Wiegold)

Let S, T be two infinite semigroups. $S \times T$ is finitely generated if and only if S and T are finitely generated and neither has indecomposable elements, in which case

$$S = \langle A \times B \rangle$$

for some finite sets A and B.

Corollary

Either $d(S^n) = \infty$ for all $n \geq 2$ or else $\mathbf{d}(S)$ is sub-exponential.

Linear – exponential – logarithmic

Theorem (Hyde, Loughlin, Quick, NR, Wallace)

Let S be a finitely generated semigroup. If S is a principal left and right ideal then $\mathbf{d}(S)$ is sub-linear, otherwise it is super-exponential.

Conjecture (Hyde)

The $\mathbf{d}$-sequence of a semigroup cannot be strictly between logarithmic and linear.

Polycyclic monoid

Definition

$$P_k = \langle b_i, c_i \ (i = 1, \ldots, k) \mid b_i c_i = 1,\ b_i c_j = 0 \ (i \neq j) \rangle$$

Fact

P_k ($k \geq 2$) is an infinite, congruence-free monoid.

Theorem (Hyde, Loughlin, Quick, NR, Wallace)

$\mathbf{d}(P_k) = (2k-1, 3k-1, 4k-1, \ldots)$.

Recursive functions

Theorem (Hyde, Loughlin, Quick, NR, Wallace)

For the monoid $R_{\mathbb{N}}$ of all partially recursive functions in one variable we have

$$\mathbf{d}(R_{\mathbb{N}}) = (2, 2, 2, \ldots).$$

Some More Open Problems

- Does there exist a semigroup (or any algebraic structure) such that $\mathbf{d}(S)$ is eventually constant, but stabilises later than the 2nd term?
- Does there exist a semigroup (or any algebraic structure) such that $\mathbf{d}(S)$ is eventually constant but with value different from $d(S)$ or $d(S)+1$?
- Is it true that the **d**-sequence of a finite algebraic structure is either logarithmic, polynomial or exponential?
- If one considers generation modulo the diagonal

$$\Delta_n(A) = \{(a, \ldots, a) \; : \; a \in A\}$$

(so that infinitely generated structures can be included too), what new (if any) growth rates appear?

. . . answer?

I could not make much sense of your maths friend's explanations. It is no matter: the masterpiece that needed it is five fathoms under. Thank you (. . .) all the same. (S. Beckett to M. Peron, two weeks later)