A relational localisation theory for topological algebras

Friedrich Martin Schneider

Technische Universität Dresden

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- explore the developed concepts for **modules of compact rings**.

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$$\mathrm{cO}_X^{(n)} := C(X^n; X), \qquad \mathrm{cR}_X^{(m)} := \{\varrho \subseteq A^m \mid \varrho \text{ closed in } X^m\},$$

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For $f \in \mathrm{O}_A^{(n)}$ and $\varrho \in \mathrm{R}_A^{(m)}$,

$$\begin{aligned} f \rhd \varrho \quad &:\Longleftrightarrow \quad \forall r_0, \ldots, r_{n-1} \in \varrho : f \circ \langle r_0, \ldots, r_{n-1} \rangle \in \varrho \\ &\Longleftrightarrow \quad \varrho \in \mathrm{Sub}(\langle A; f \rangle^m) \\ &\Longleftrightarrow \quad f \in \mathrm{Hom}(\langle A; \varrho \rangle^n; \langle A; \varrho \rangle). \end{aligned}$$

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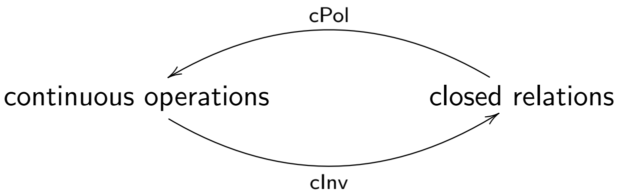
How can we describe the closure system induced by this Galois connection?



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A set $F \subseteq \mathrm{O}_A$ is called clone of operations on A if

(1) F contains all projections,

(2) for $m, n \in \mathbb{N}$, $f \in F^{(n)}$, $f_0, \ldots, f_{n-1} \in F^{(m)}$, we also have $f \circ \langle f_0, \ldots, f_{n-1} \rangle \in F$.

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Obviously, cO_X is a clone of operations on A.

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$$\overline{\bigwedge\nolimits_{(\varphi_i)_{i\in I}}^{\varphi,Y,X} (\varrho_i)_{i\in I}}^{X^m} \in Q,$$

where

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$$\begin{aligned}
\text{s-Loc}\, F &:= \{f \in \mathrm{cO}_X^{(n)} \mid n \in \mathbb{N}, \forall a \in (A^n)^s, U \in T^s : \\
&\quad [f(a_0) \in U_0, \ldots, f(a_{s-1}) \in U_{s-1}] \Rightarrow \\
&\quad [\exists g \in F : g(a_0) \in U_0, \ldots, g(a_{s-1}) \in U_{s-1}]\}, \\
\text{s-LOC}\, Q &:= \{\varrho \in \mathrm{cO}_X \mid \forall \sigma \subseteq \varrho, |\sigma| \leq s : \exists \varrho' \in Q : \sigma \subseteq \varrho' \subseteq \varrho\}, \\
\mathrm{Loc}\, F &:= \bigcap_{s \in \mathbb{N}} \text{s-Loc}\, F, \\
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What are the suitable subsets for this kind of localisation theory?

Finding suitable subsets

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Let $\mathbf{A} = \langle A; T; F \rangle$ be a topological algebra. For $U \subseteq A$,

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Additionally, if (a) holds, then

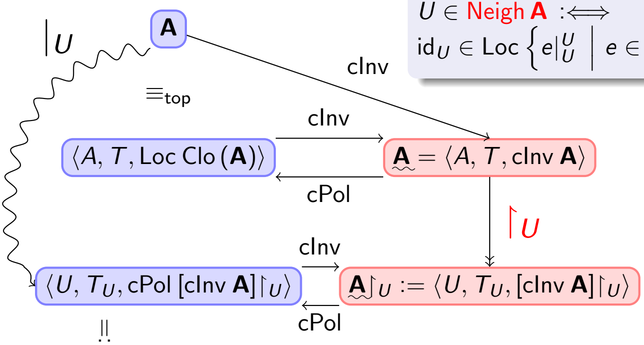
$$[Q]\restriction_U := \{\varrho\restriction_U \mid \varrho \in Q\} \text{ is locally closed}$$

for every locally closed clone of closed relations $Q \subseteq \operatorname{cInv}\mathbf{A}$.

Restricting algebras to neighbourhoods

Definition (neighbourhood)

$$U \in \text{Neigh}\,\mathbf{A} \;:\Longleftrightarrow\; \mathrm{id}_U \in \mathrm{Loc}\left\{ e|_U^U \;\middle|\; e \in \mathrm{Loc}\,\mathrm{Clo}^{(1)}(\mathbf{A}),\, \mathrm{im}\, e \subseteq U \right\}$$



$$\mathbf{A}|_U = \left\langle U, T_U, \mathrm{Loc}\left\{ f|_{U^{\mathrm{ar}\,f}}^U \;\middle|\; f \in \mathrm{Loc}\,\mathrm{Clo}(\mathbf{A}),\, \mathrm{im}\, f \subseteq U \right\}\right\rangle$$

$$= \left\langle U, T_U, \mathrm{Loc}\left\{ f|_{U^{\mathrm{ar}\,f}}^U \;\middle|\; f \in \mathrm{Loc}\,\mathrm{Clo}(\mathbf{A}),\, f \rhd U \right\}\right\rangle$$

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Let $\mathcal{U} \subseteq \text{Neigh}\,\mathbf{A}$.

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Moreover, let

$$E_{\mathbf{A}}(\mathcal{U}) := \bigcup \{E_{\mathbf{A}}(U) \mid U \in \mathcal{U}\}.$$

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(b) $\mathrm{id}_A \in \operatorname{Loc} \langle E_{\mathbf{A}}(\mathcal{U}) \rangle_{\mathbf{A}^A}$.

(c) *There is an index set* Φ *and a map* $\underset{\sim}{\mathbf{B}} : \Phi \to \{ \underset{\sim}{\mathbf{A}}\restriction_U \mid U \in \mathcal{U} \}$ *such that* $\underset{\sim}{\mathbf{A}}$ *is approximately a retract of* $\prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{B}}(\varphi)$, *i.e. there exists* $M : \underset{\sim}{\mathbf{A}} \to \prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{B}}(\varphi)$ *with*

$$\mathrm{id}_A \in \operatorname{Loc} \left\{ \Lambda \circ M \;\middle|\; \Lambda : \prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{B}}(\varphi) \to \underset{\sim}{\mathbf{A}} \right\}.$$

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Reminder

Let $\mathbf{R} = \langle R, +, -, \cdot, 0 \rangle$ be a ring.

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Theorem (Gabriel, 1962)

Let $\mathbf{R} = \langle R, S, +, -, \cdot, 0, 1 \rangle$ be a compact Hausdorff topological ring, $0 \neq 1$. Then there exists an orthogonal set $E \subseteq \operatorname{Id} \mathbf{R}$ of primitive idempotents such that $1 = \sum_{e \in E} e$.

Seriously, what about the example?

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(i) *Every c-cover of* $\mathbf{M}|_U$ *contains* U*.*

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Let $U \in \mathrm{cNeigh}\,\mathbf{M}$, $|U| > 1$, *and* $m \in \mathbb{N}$, $\varrho, \sigma \in \mathrm{cInv}^{(m)}\,\mathbf{M}$ *such that* $\varrho\restriction_U \neq \sigma\restriction_U$. *TFAE:*

(i) *Every c-cover of* $\mathbf{M}|_U$ *contains* U.

(ii) $U \in \mathrm{Min}_{\subseteq}((\mathrm{cNeigh}\,\mathbf{M}) \setminus \{\{0\}\})$.

Seriously, what about the example?

$\mathbf{R} = \langle R, S, +, -, \cdot, 0, 1\rangle$ compact Hausdorff topological ring,
$\mathbf{M} = \langle M, T, +, -, 0, (\lambda(r))_{r\in R}\rangle$ Hausdorff topological $\mathbf{R}$-module.

Lemma

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(iv) *There exists a primitive idempotent* $e \in \operatorname{Id} \mathbf{R}$ *such that* $U = \operatorname{im} \lambda(e)$.

The very last slide

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Thank you for your attention!!

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