Kueker's conjecture

Predrag Tanović

Novi Sad, 2012

- L is a countable language (function, relation, constant symbols).
- T is a complete, first-order L-theory with infinite models. (T is *complete* if for any L-sentence φ:

$$\text{either } T \models \varphi \text{ or } T \models \neg\varphi.)$$

- L is a countable language (function, relation, constant symbols).
- T is a complete, first-order L-theory with infinite models. (T is *complete* if for any L-sentence φ:

$$\text{either } T \models \varphi \text{ or } T \models \neg\varphi.)$$

Definition

Let κ be an infinite cardinal. T is *κ-categorical* (or categorical in κ) if it has a unique, up to isomorphism, model of size κ.

$$c(T) = \{\kappa \mid T \text{ is } \kappa\text{-categorical }\}$$

For most T's $c(T) = \emptyset$.

The following was conjectured by Los in 1954 and proved by Morley in 1965.

Theorem

If T is categorical in one uncountable power then it is categorical in all uncountable powers.

Examples

$c(T) = \{\aleph_0\}$

- The theory of $(Q, <)$.
- Theory of the random graph.

Totally categorical: $c(T) = \{\kappa \mid \kappa \geq \aleph_0\}$

- Theory of an infinite set.
- Theory of an infinite vector space over a finite field F; the language is $(L = \{+, 0\} \cup \{f \cdot \mid f \in F\})$.

$c(T) = \{\kappa \mid \kappa \geq \aleph_1\}$

- Theory of a vector space over a (countably) infinite field.
- Theory of algebraically closed fields of a fixed characteristic.

There is a natural notion of dimension in models of an uncountably categorical theory: every model is uniquely determined by its 'ϕ-dimension' where $\phi(x)$ is a strongly minimal formula.

Definition

$\phi(x)$ is *strongly minimal* if for any $M \models T$ the only definable (with parameters from M) subsets of $\phi(M)$ are finite and co-finite ones.

Definition

$M \models T$ is $\aleph_0$-*saturated* if for all $\bar{a} \in M^{<\omega}$: whenever $\{\phi_i(x, \bar{a}) \mid i \in \omega\}$ is finitely satisfied in M then it is satisfied in M.

Remark

If T is categorical in some infinite power then:

every uncountable model of T is $\aleph_0$-saturated.

Definition

$M \models T$ is $\aleph_0$-*saturated* if for all $\bar{a} \in M^{<\omega}$: whenever $\{\phi_i(x, \bar{a}) \mid i \in \omega\}$ is finitely satisfied in M then it is satisfied in M.

Remark

If T is categorical in some infinite power then:

every uncountable model of T is $\aleph_0$-saturated.

Kueker's conjecture

If every uncountable model of T is $\aleph_0$-saturated then T is categorical in some infinite power.

Definition

$M \models T$ is *$\aleph_0$-saturated* if for all $\bar{a} \in M^{<\omega}$: whenever $\{\phi_i(x, \bar{a}) \mid i \in \omega\}$ is finitely satisfied in M then it is satisfied in M.

Remark

If T is categorical in some infinite power then:

every uncountable model of T is $\aleph_0$-saturated.

Kueker's conjecture

If every uncountable model of T is $\aleph_0$-saturated then T is categorical in some infinite power.

Hrushovski in 1986. proved that Kueker's conjecture is true for:

- T stable.
- T interpreting a linear order.

Definition

T is *unstable* if there is $M \models T$, (n and) $\{\bar{a}_i \in M^n \mid i \in \omega\}$ and $\phi(\bar{x}, \bar{y})$ such that:

$$M \models \phi(\bar{a}_i, \bar{a}_j) \text{ iff } i < j$$

Uncountably categorical theories are stable.

Theorem(Shelah)

If T is unstable then there is $M \models T$ such that at least one of the following two holds:

(1) The independence property (IP)
There are n, $A = \{\bar{a}_i \in M^n \mid i \in \omega\}$ and $\phi(\bar{x}, \bar{y})$ such that:

$$(A, \phi) \text{ is isomorphic to the random graph}$$

(2) The strict order property (SOP)
There is (n and) a definable partial order on M^n having infinite strictly increasing chains.

The remaining parts

Definition

$(M, ...)$ is *almost minimal* if:
(1) for any L-formula $\phi(x)$ either $\phi(M)$ or $\neg\phi(M)$ is finite; and
(2) there are infinitely many disjoint finite subsets of the form $\phi(M)$ as above.

The remaining parts

Definition

$(M, ...)$ is *almost minimal* if:
(1) for any L-formula $\phi(x)$ either $\phi(M)$ or $\neg\phi(M)$ is finite; and
(2) there are infinitely many disjoint finite subsets of the form $\phi(M)$ as above.

The remaining parts of the conjecture:
(A) The case $T = Th(M, ...)$ where M is almost minimal.
(B) T without SOP but with IP.