Local monotonicities and lattice derivatives of Boolean and pseudo-Boolean functions

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Partial derivatives

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- The partial derivative of $f\colon \{0,1\}^n \to \mathbb{R}$ w.r.t. x_k is the function $\Delta_k f\colon \{0,1\}^n \to \mathbb{R}$ defined by

$$\begin{aligned}\Delta_k f(\mathbf{x}) &= f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0)\\ &= f\left(x_1, \ldots, 1, \ldots, x_n\right) - f\left(x_1, \ldots, 0, \ldots, x_n\right).\end{aligned}$$

Observe that $\Delta_k f$ does not depend on x_k.

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Example

The partial derivatives of the Boolean sum $f\left(x_1, x_2\right) = x_1 \oplus x_2 = x_1 + x_2 - 2x_1x_2$ are

$$\begin{aligned}\Delta_1 f(x_1, x_2) &= f(1, x_2) - f(0, x_2) = 1 - 2x_2,\\ \Delta_2 f(x_1, x_2) &= f(x_1, 1) - f(x_1, 0) = 1 - 2x_1.\end{aligned}$$

Monotonicity

- f is isotone (positive, order-preserving, nondecreasing) in x_k if

$$\Delta_k f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \{0,1\}^n .$$

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- f is monotone in x_k if it is either isotone or antitone in x_k, i.e., if $\Delta_k f(\mathbf{x})$ does not change sign.

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- f is monotone (isotone, antitone) if it is monotone (isotone, antitone) in all of its variables.
- All unary functions are monotone.
- The only non-monotone binary Boolean functions are $x_1 \oplus x_2$ and $x_1 \oplus x_2 \oplus 1$.

Local monotonicities

Definition

We say that $f\colon \{0,1\}^n \to \mathbb{R}$ is *p-locally monotone* if, for every $k \in [n]$ and every $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, we have

$$\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| < p \quad \Rightarrow \quad \Delta_k f(\mathbf{x}) \Delta_k f(\mathbf{y}) \geq 0.$$

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- Every function is 1-locally monotone.

Theorem

A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is 2-locally monotone iff

$$\left| \Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y}) \right| \leq \sum_{i \in [n] \setminus \{k\}} |x_i - y_i|.$$

Lattice derivatives

We define the partial lattice derivatives of $f\colon \{0,1\}^n \to \mathbb{R}$ w.r.t. x_k by

$$\wedge_k f\colon \{0,1\}^n \to \mathbb{R},\ \wedge_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1) = \min\left(f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\right),$$
$$\vee_k f\colon \{0,1\}^n \to \mathbb{R},\ \vee_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1) = \max\left(f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\right).$$

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$$\wedge_1 f(x_1, x_2) = f(1, x_2) \wedge f(0, x_2) = (1 \oplus x_2) \wedge x_2 = 0,$$
$$\vee_1 f(x_1, x_2) = f(1, x_2) \vee f(0, x_2) = (1 \oplus x_2) \vee x_2 = 1.$$

The second-order lattice derivatives are

$$\vee_2 \wedge_1 f(x_1, x_2) = \vee_2 0 = 0,$$
$$\wedge_1 \vee_2 f(x_1, x_2) = \wedge_1 1 = 1.$$

Permutable lattice derivatives

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A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ *is 2-locally monotone iff*

$$\vee_k \wedge_j f = \wedge_j \vee_k f \text{ for all } j \neq k.$$

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We say that $f\colon \{0,1\}^n \to \mathbb{R}$ has p-permutable lattice derivatives, if

$$O_{k_1} \cdots O_{k_p} f = O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f$$

holds for every p-element set $\{k_1, \ldots, k_p\} \subseteq \{1, \ldots, n\}$, for all operators $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ and for every permutation $\pi \in S_p$.

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Theorem

If a function has p-permutable lattice derivatives, then it has $(p-1)$*-permutable lattice derivatives.*

Local monotonicities vs. permutable lattice derivatives

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Let $f\colon \{0,1\}^n \to \{0,1\}$ be the function that takes the value 0 on all tuples of the form

$$(\overbrace{1,\ldots,1}^{m},0,\ldots,0) \text{ with } 0 \leq m \leq n,$$

and takes the value 1 everywhere else. Then f has n-permutable lattice derivatives, but it is only 2-locally monotone.

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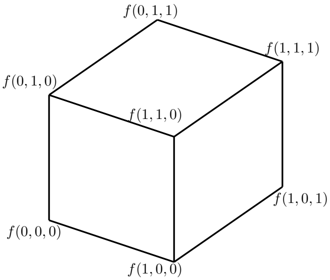
Theorem

For symmetric functions, p-local monotonicity is equivalent to p-permutability of lattice derivatives.

Sections

A section of a function f is any function g that can be obtained from f by substituting constants to some of the variables of f.

For example, if $f\colon \{0,1\}^3 \to \mathbb{R}$, then
$g\colon \{0,1\}^2 \to \mathbb{R},\ g(x_1,x_2) := f(x_1,x_2,0)$ is a section of f.



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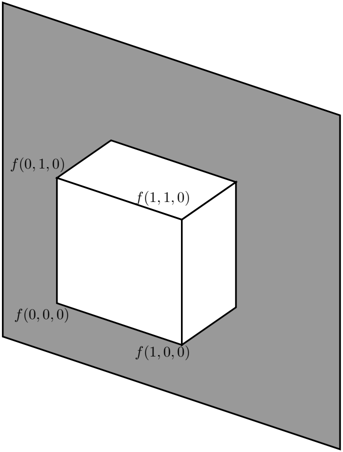

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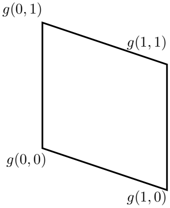

Forbidden sections

Theorem

If a function is nice, then all of its sections are also nice, where "nice" stands for any of the previously discussed properties.

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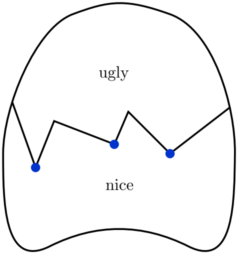

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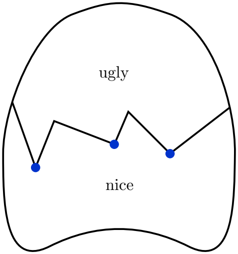

Corollary

A function is nice iff none of the minimal ugly functions appear among its sections.

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A Boolean function is isotone iff $x_1 \oplus 1$ *does not appear among its sections.*

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Theorem

A Boolean function is 2-locally monotone iff neither $x_1 \oplus x_2$ *nor* $x_1 \oplus x_2 \oplus 1$ *appears among its sections.*

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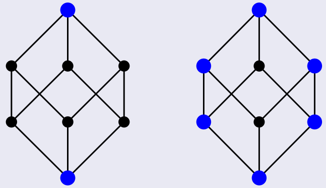
A Boolean function is 2-locally monotone iff neither $x_1 \oplus x_2$ nor $x_1 \oplus x_2 \oplus 1$ appears among its sections.

Conjecture

A Boolean function has permutable lattice derivatives iff none of the following functions appear among its sections:



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M. Couceiro, J.-L. Marichal, T. Waldhauser, *Locally monotone Boolean and pseudo-Boolean functions*, to appear in Discrete Applied Mathematics, arXiv:1107.1161.

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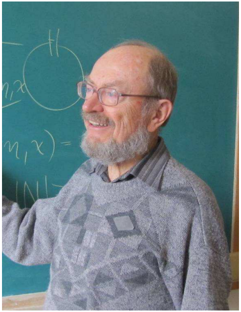

Dedicated to the 80th birthday of Béla Csákány