Chains of subsemigroups

J. D. Mitchell

Joint work with: P. J. Cameron, M. Gadouleau, Y. Péresse

School of Mathematics and Statistics, University of St Andrews

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University
of
St Andrews

An email from Attila

12th of August 2013 at 08:34:

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where $b(n)$ is the number of ones in the binary expansion of n (!!).

The length of a semigroup

Let S be a semigroup. A collection of subsemigroups of S is called a *chain* if it is totally ordered with respect to inclusion. For example, if

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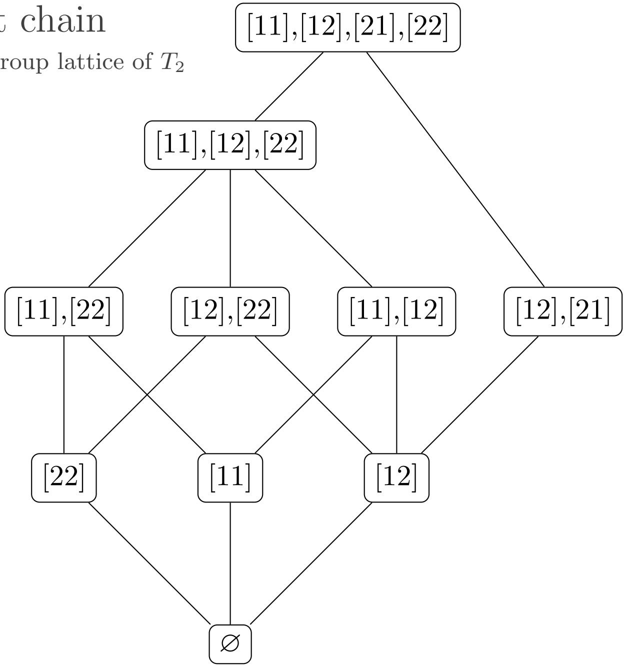
The *length* $l(S)$ of a semigroup S is the largest number of non-empty subsemigroups of S in a chain minus 1.

A longest chain

The subsemigroup lattice of T_2



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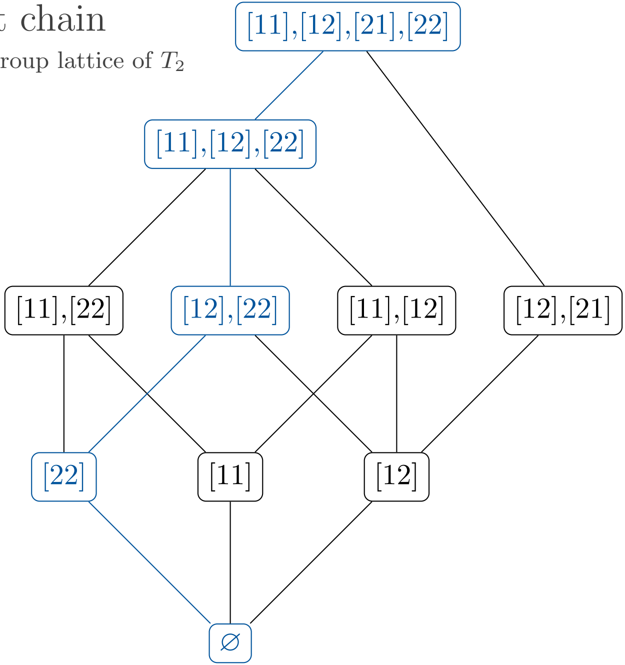

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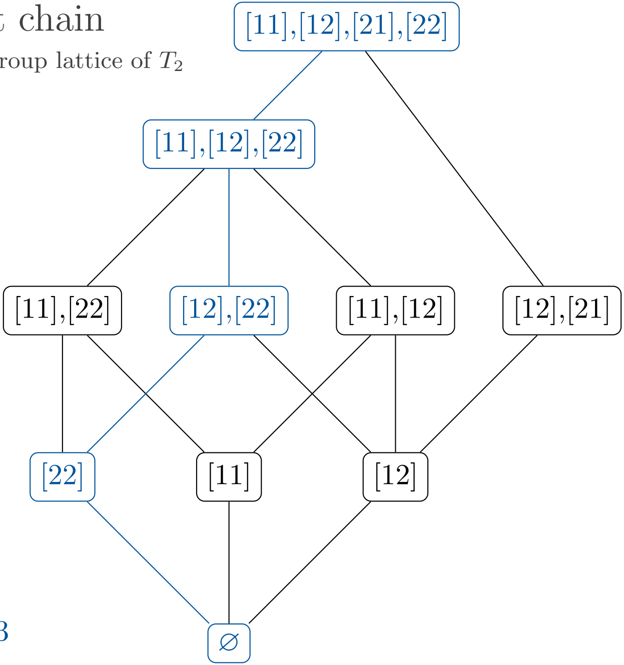

$\Rightarrow l(T_2) = 3$

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  in $b(n) - 1$ steps
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- $S_{2^t} > S_{2^{t-1}} \wr S_2 > S_{2^{t-1}} \times S_{2^{t-1}} > \cdots > \mathbf{1}$ for $t > 0$
- Then do the bookkeeping.

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The Classification of Finite Simple Groups is needed to show that there is no longer chain.

For some values of n (e.g. 15), there are other chains of the same length.

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The question of finding $l(S_n)$ was first raised by László Babai in the context of computational group theory.

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We regard a formula containing $l(G)$ for some group G as "known".

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Analogues of Cayley's theorem states that every finite semigroup is isomorphic to a subsemigroup of some T_n, and that every finite inverse semigroup is isomorphic to an inverse subsemigroup of some I_n.

Subsemigroups and quotients

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None of these facts is very useful for us!

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Proposition (cf. Ganyushkin-Livinsky '11)

Let S be a semigroup and let I be an ideal of S. Then

$$l(S) = l(I) + l(S/I).$$

Green's relations

If S is a semigroup and $x, y \in S$, then we write

- $x \mathscr{L} y$ if $S^1 x = S^1 y$
- $x \mathscr{R} y$ if $x S^1 = y S^1$
- $x \mathscr{J} y$ if $S^1 x S^1 = S^1 y S^1$
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These relations are equivalences called *Green's relations*, and their classes are *Green's classes*.

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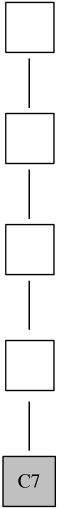

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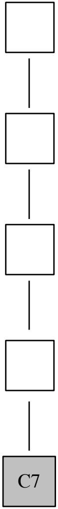

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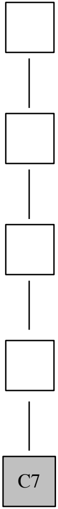

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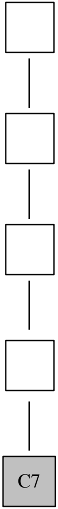
Proof

Repeatedly apply the $l(S) = l(I) + l(S/I)$ lemma:
$l(S) = m + l(C_n) - 1$ and $l(C_n) = \Omega(n)$. $\square$

Monogenic semigroups

Let S be a semigroup generated by a single element s where 5 and 7 are the least numbers such that $s^{5+7} = s^5$. The $\mathscr{J}$-classes of S ordered by containment of ideals look like $\longrightarrow$



Proposition

Let S be a semigroup generated by a single element s and let $m, n \in \mathbb{N}$ be the least numbers such that $s^{m+n} = s^m$. Then $l(S) = m + \Omega(n) - 1$, where $\Omega(n)$ is the number of prime power divisors of n.

Proof

Repeatedly apply the $l(S) = l(I) + l(S/I)$ lemma:
$l(S) = m + l(C_n) - 1$ and $l(C_n) = \Omega(n)$. □

So, in the example, $l(S) = 5 + 1 - 1 = 5$.

Principal factors

The *principal factor* J^* of a $\mathscr{J}$-class J is the set $J \cup \{0\}$ with multiplication

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Lemma

Let S be a finite regular semigroup and let $J_1, J_2, \ldots, J_m$ be the $\mathscr{J}$-classes of S. Then

$$l(S) = l(J_1^*) + l(J_2^*) + \cdots + l(J_m^*) - 1.$$

Inverse semigroups

An *inverse semigroup* is a semigroup S such that for all $x \in S$, there exists a unique $x^{-1} \in S$ where $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

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Theorem (cf. Ganyushkin and Livinsky (2011))

Let S be a finite inverse semigroup with $\mathscr{J}$-classes $J_1, \ldots, J_m$. If $n_i \in \mathbb{N}$ denotes the number of $\mathscr{L}$- and $\mathscr{R}$-classes in J_i, and G_i is any maximal subgroup of S contained in J_i, then

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$$\begin{aligned} l(S) &= -1 + \sum_{i=1}^{m} l(J_i^*) \\ &= -1 + \sum_{i=1}^{m} n_i(l(G_i) + 1) + \frac{n_i(n_i - 1)}{2}|G_i| + (n_i - 1). \end{aligned}$$

The symmetric inverse monoid, part I

The *symmetric inverse monoid* I_n consists of all bijections between subsets of $X = \{1, \ldots, n\}$.

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If $f, g \in I_n$, then

- $f \mathscr{L} g$ if and only if $\text{im}(f) = \text{im}(g)$;
- $f \mathscr{R} g$ if and only if $\text{dom}(f) = \text{dom}(g)$;
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$$l(I_n) = -1 + \sum_{i=1}^{n} \binom{n}{i} (l(S_i) + 1) + \frac{\binom{n}{i}(\binom{n}{i} - 1)}{2} |S_i| + \binom{n}{i} - 1.$$

The symmetric inverse monoid, part II

n	1	2	3	4	5	6	7	8
$\|I_n\|$	2	7	34	209	1 546	13327	130 922	1 441 729
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The same limit holds for various other well-known inverse semigroups: the dual symmetric inverse monoid, the semigroup of partial order-preserving injective mappings, and so on.

The full transformation monoid

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Here are the first few values:

n	2	3	4	5	6	7	8
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We don't know if $l(T_n)/|T_n|$ tends to a limit as $n \to \infty$.

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Theorem

The number of subsemigroups of T_n is at least $2^{cn^{n-1/2}}$ where

$$c = \frac{\mathrm{e}^{-2}}{3\sqrt{3(\mathrm{e}^{-1} - 2\mathrm{e}^{-2})}}.$$

Note that this is a bit less that $2^{c|T_n|}$ (because of the $-1/2$ in the exponent).

Minimum number of generators

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The smallest number $d(n)$ such that any subsemigroup of T_n can be generated by $d(n)$ elements is at least $(c - o(1))n^{n-1/2}$ where c is the constant in the previous theorem.

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Jerrum gave a weaker bound of $n - 1$ but with an algorithmic proof. Given a sequence of elements of S_n there is a polynomial time algorithm that produces at most $n - 1$ elements generating the same group.

An (impractical) algorithm for finding the length

The *principal factor* J^* of a $\mathscr{J}$-class J is the set $J \cup \{0\}$ with multiplication

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$$\begin{aligned} l(S) &= -1 + \sum_{i=1}^{m} l(J_i^*) \\ &= -1 + \sum_{i=1}^{m} \max\{\, l(T) \,:\, T \leq J_i^*,\ T \text{ maximal} \,\} + 1. \end{aligned}$$

The Rees Theorem

If G is a group, I and J are sets, and $P = (p_{j,i})_{j\in J, i\in I}$, then the *Rees 0-matrix semigroup* $\mathcal{M}^0[I, G, J; P]$ is the set $(I \times G \times J) \cup \{0\}$ with multiplication:

$$(i, g, j)(k, h, l) = \begin{cases} (i, gp_{j,k}h, l) & \text{if } p_{j,k} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem (Rees' Theorem)

Let S be a finite semigroup and let J be a regular $\mathscr{J}$-class of S. Then $J^* \cong \mathcal{M}^0[I, G, J; P]$ where I, J are finite sets, G is a finite group, P is a $|J| \times |I|$ matrix with entries in $G \cup \{0\}$, and every row and column of P contains a non-zero entry.

Maximal subsemigroups

Theorem (Graham-Graham-Rhodes '68)

Let $S = \mathcal{M}^0[I, G, J; P]$ *be a finite regular Rees 0-matrix semigroup, and let* M *be a maximal subsemigroup of* S*.*

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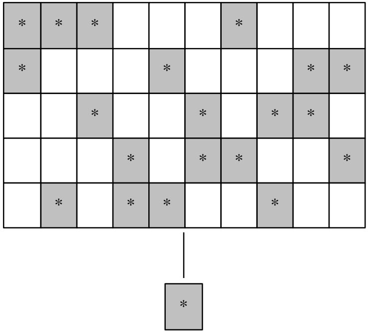
(c) $\mathcal{M}^0[I, G, J \setminus \{j\}; P]$ *for every* $j \in J$ *s.t. this semigroup is regular;*

(d) $\mathcal{M}^0[I, G, J; P] \setminus (I' \times G \times J')$ *for some* $I' = I \setminus X$, $J' = J \setminus Y$, *and* $X \times Y$ *is a maximal "rectangle" of zeros.*

An example



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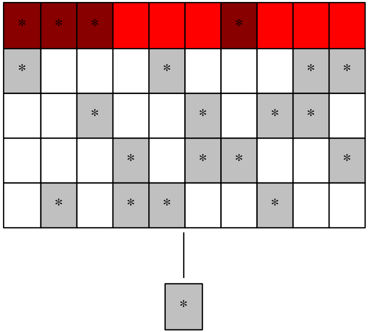

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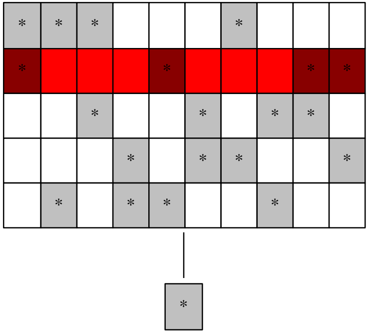

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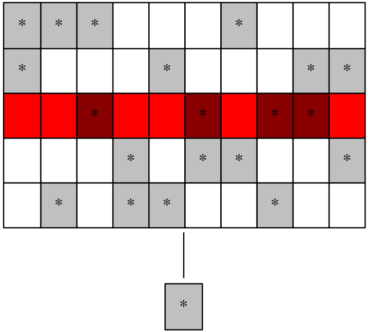

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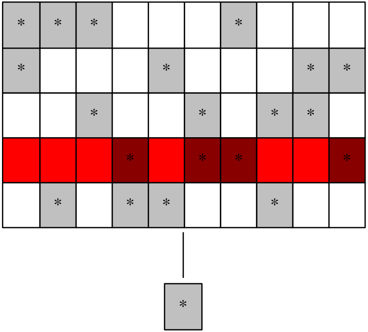

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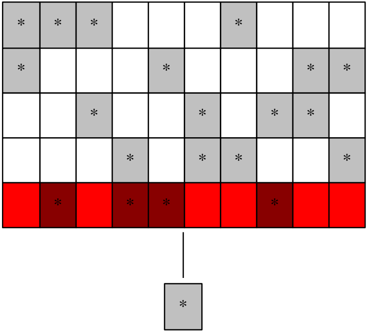

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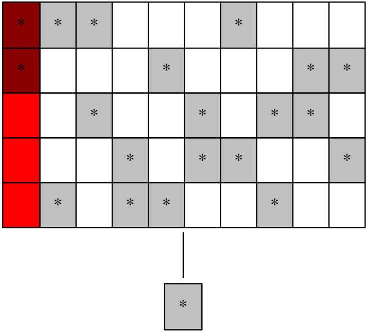

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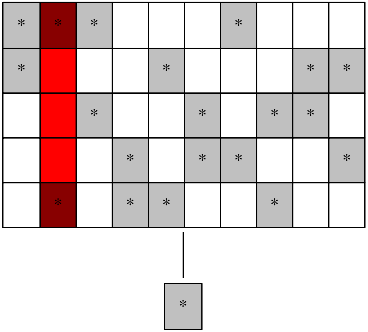

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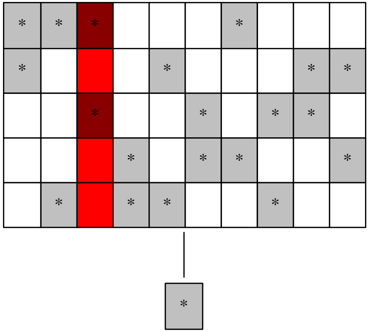

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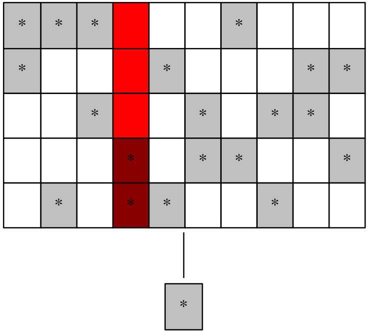

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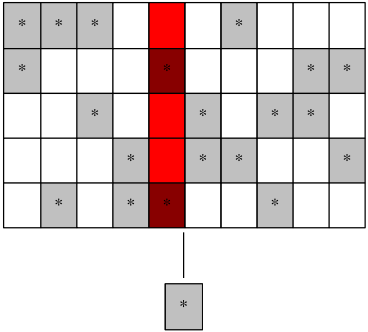

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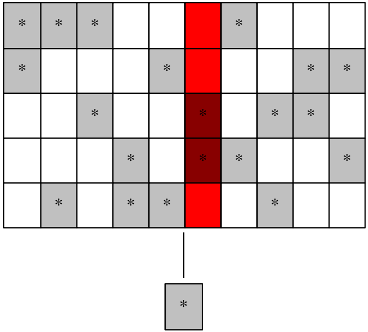

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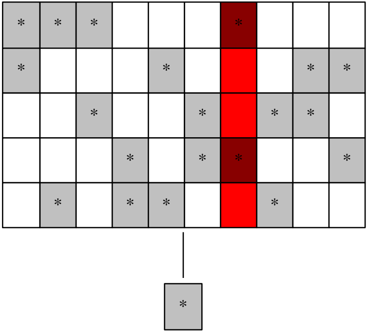

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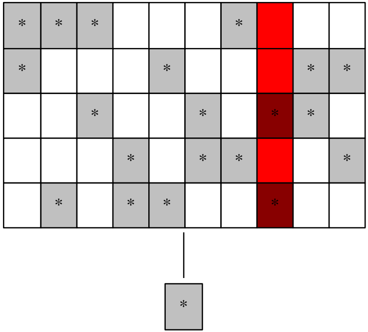

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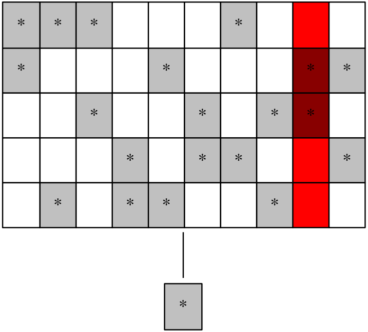

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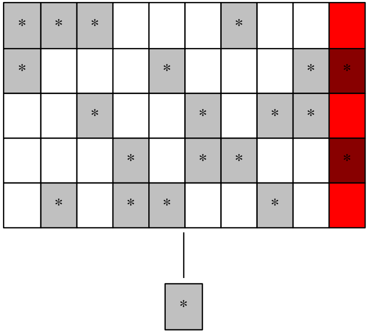

Maximal rectangles of zeros

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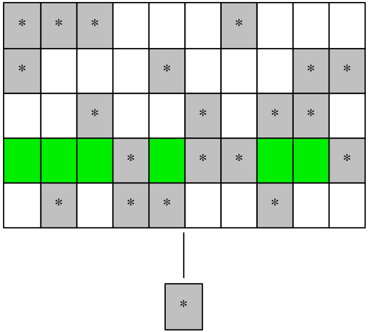

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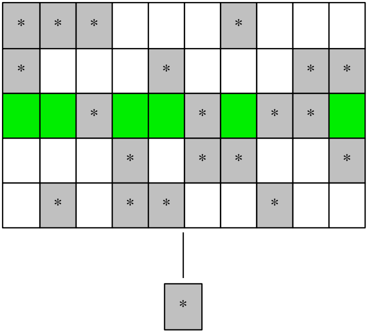

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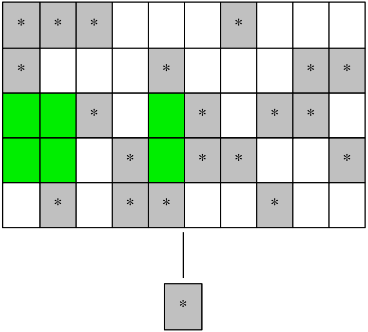

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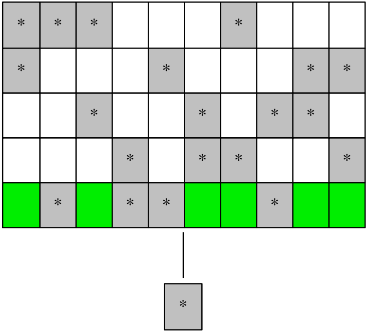

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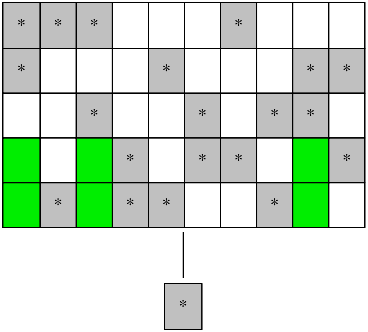

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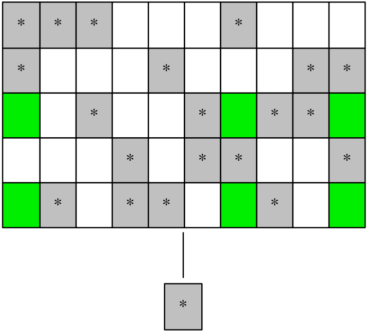

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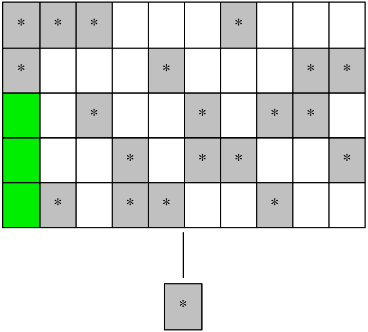

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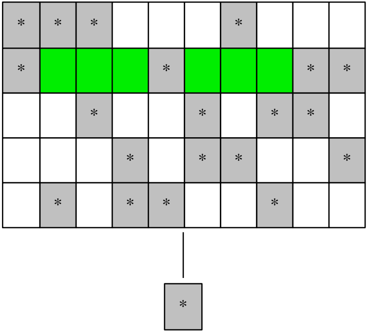

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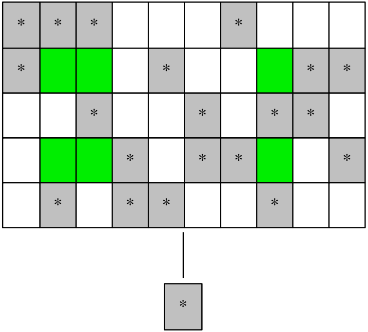

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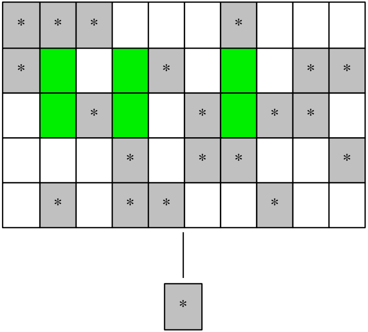

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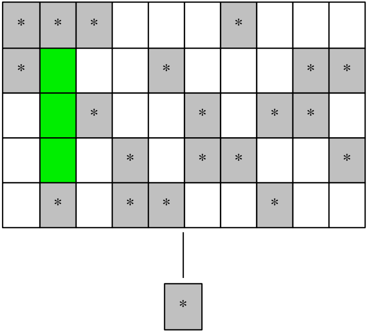

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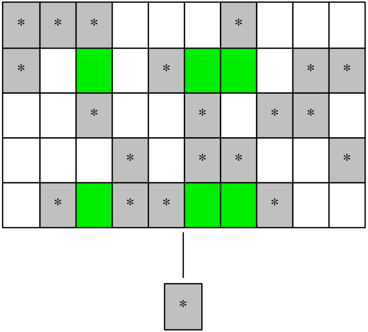

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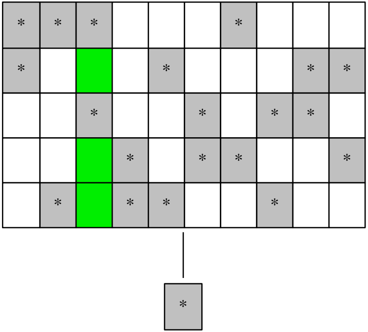

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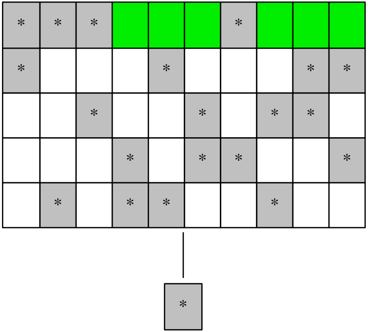

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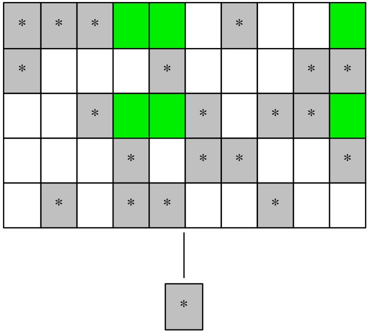

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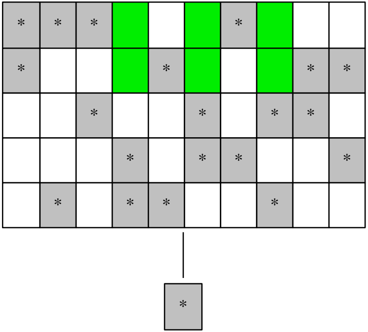

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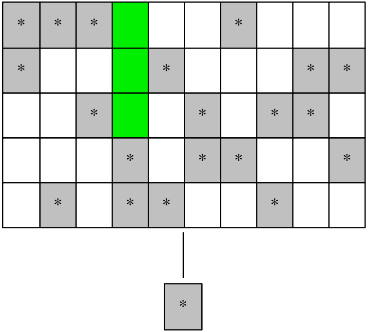

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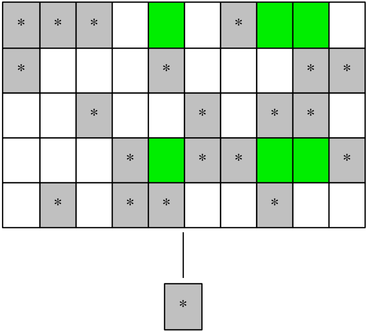

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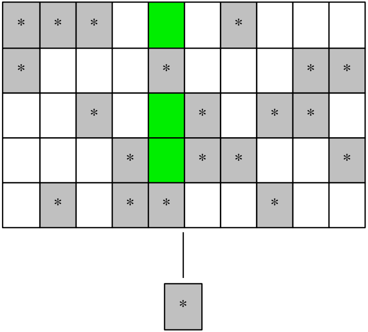

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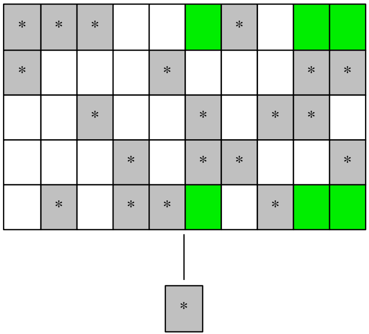

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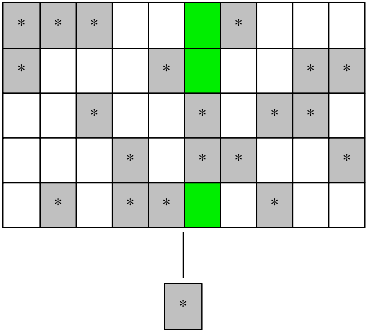

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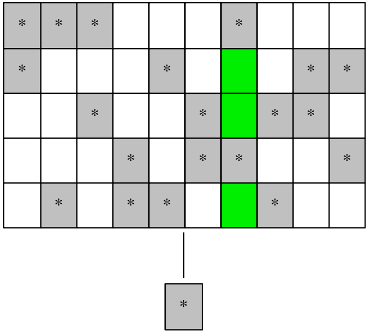

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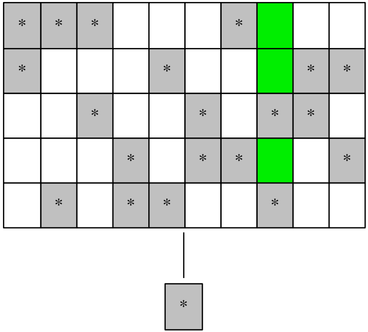

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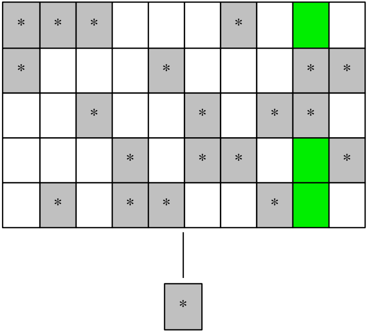

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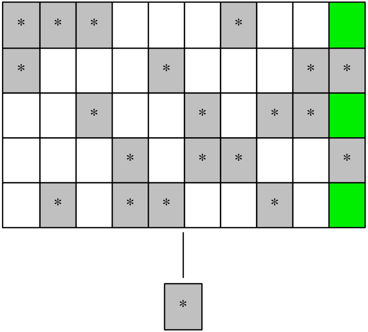
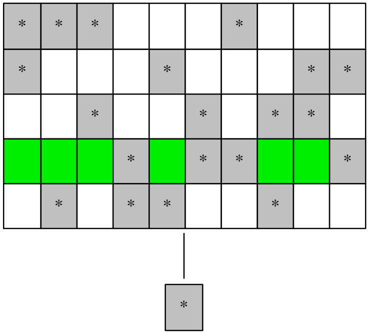


Maximal subsemigroups from maximal rectangles

$\mathcal{M}^0[I, G, J; P] \setminus (I' \times G \times J')$ for some $I' = I \setminus X$, $J' = J \setminus Y$, and $X \times Y$ is a maximal "rectangle" of zeros.



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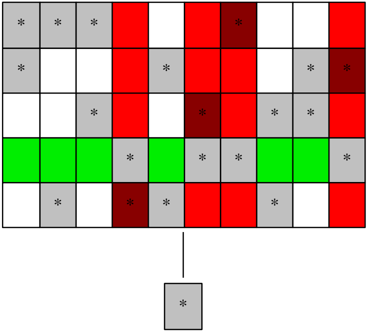

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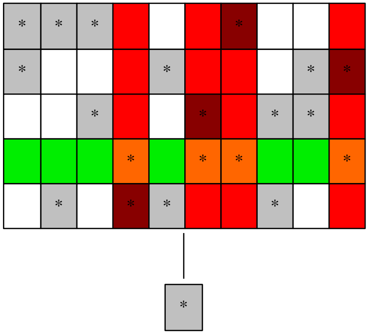

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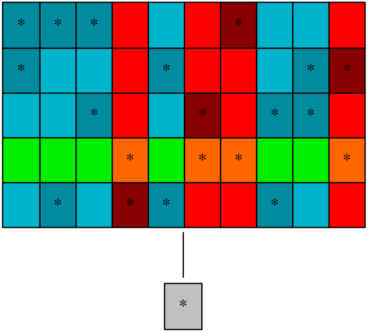

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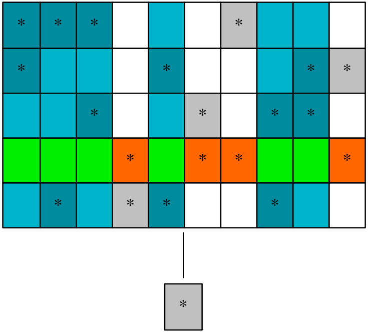

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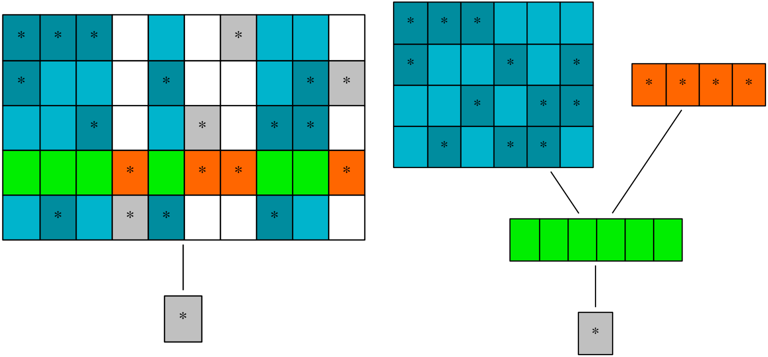

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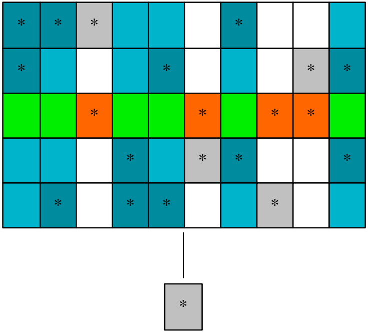

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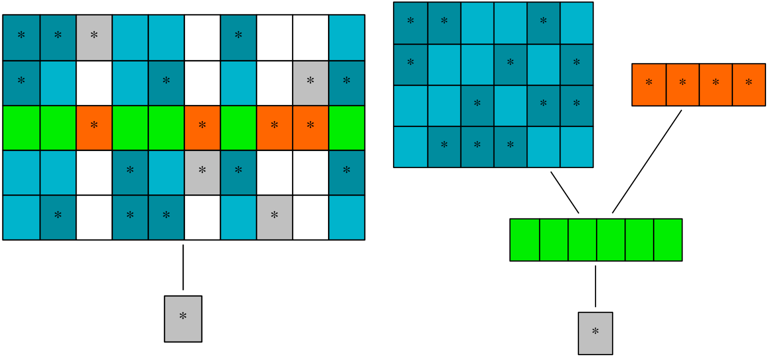

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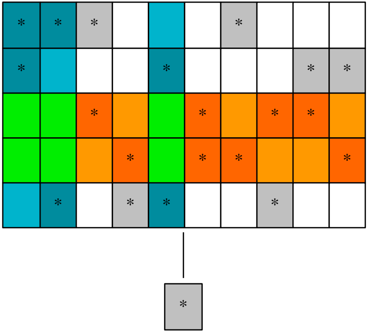

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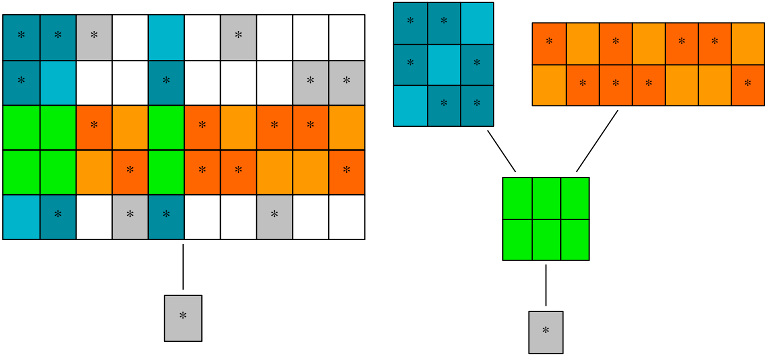

Principal factors of inverse semigroups

If G is a group, and $n \in \mathbb{N}$, then define $B(G, n)$ to be $\{1, \ldots, n\} \times G \times \{1, \ldots, n\}$ where

$$(i, g, j)(k, h, l) = \begin{cases} (i, gh, l) & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Such a $B(G, n)$ is called a *Brandt semigroup*.

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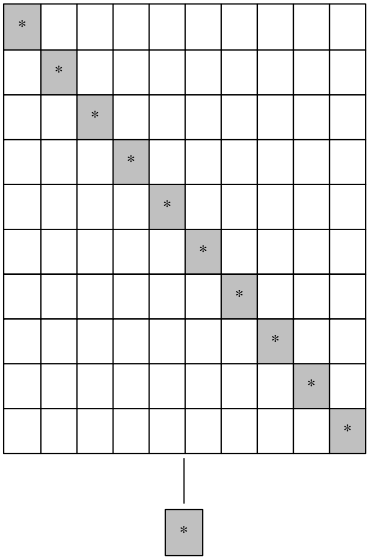
Theorem (Graham-Graham-Rhodes '68)

Let $S = B(G, n)$ be a finite Brandt semigroup, and let M be a maximal subsemigroup of S. Then M is of the form:

(a) *$B(H, n)$ where H is a maximal subgroup of G;*

(b) *$B(G, n) \setminus (I' \times G \times I'')$ for some $I' = I \setminus X$, $I'' = I \setminus Y$, and $X \times Y$ is a maximal "rectangle" of zeros.*

Can't remove a row or column



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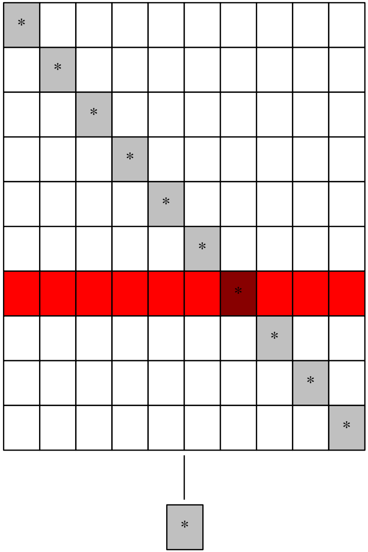

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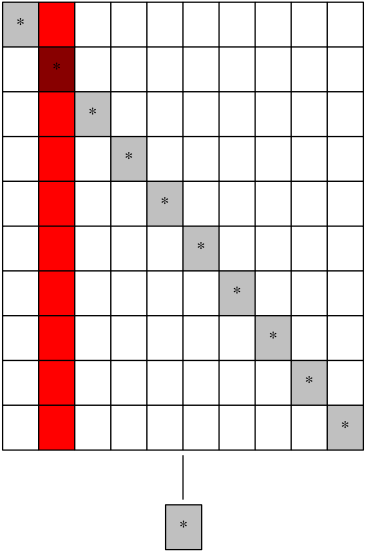

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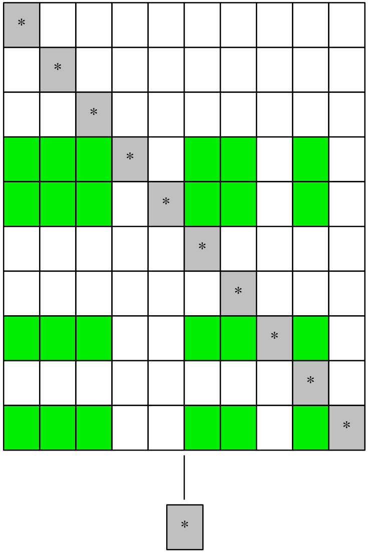

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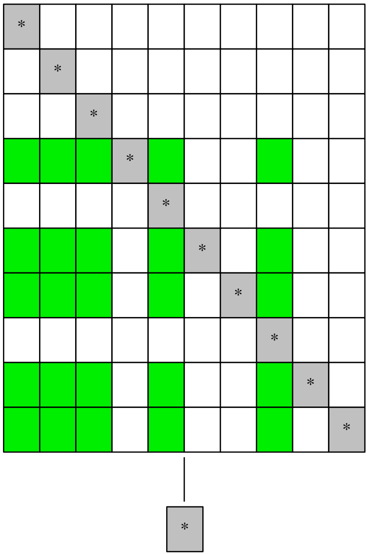

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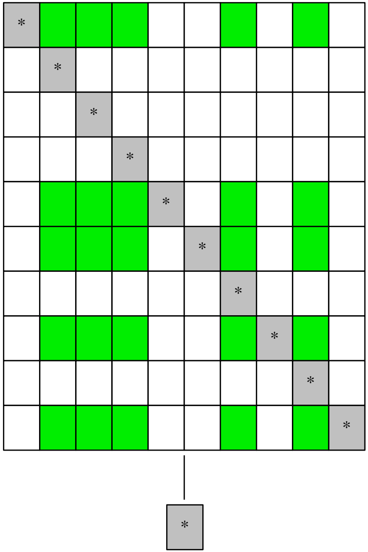

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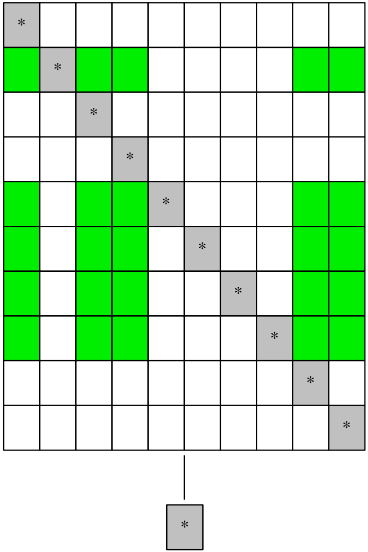

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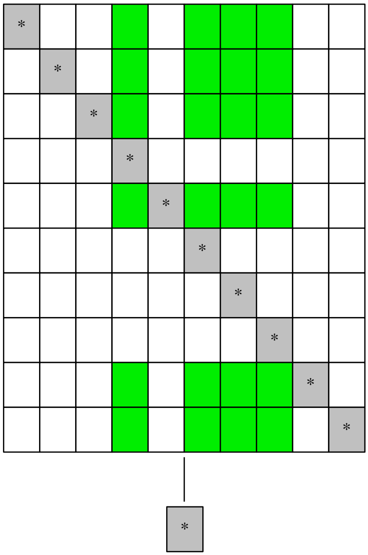

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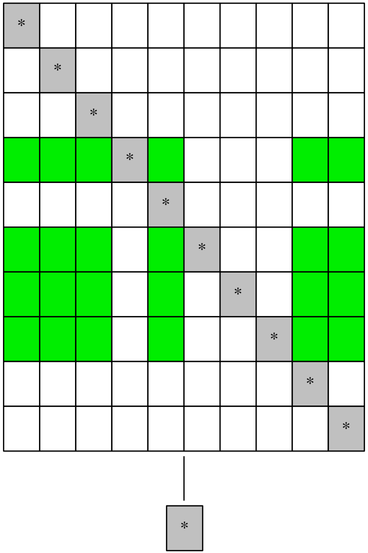

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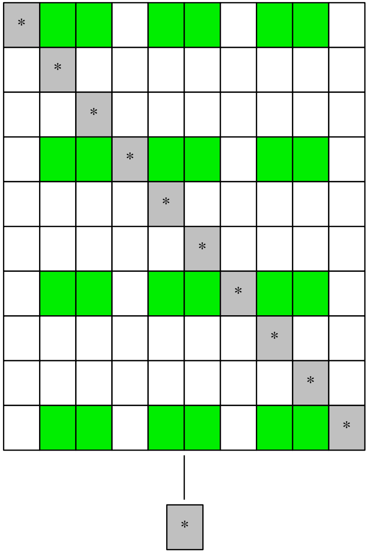

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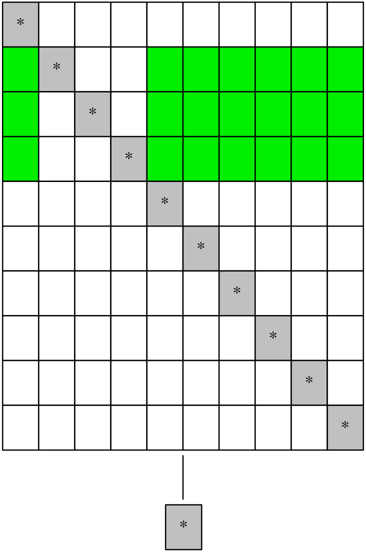

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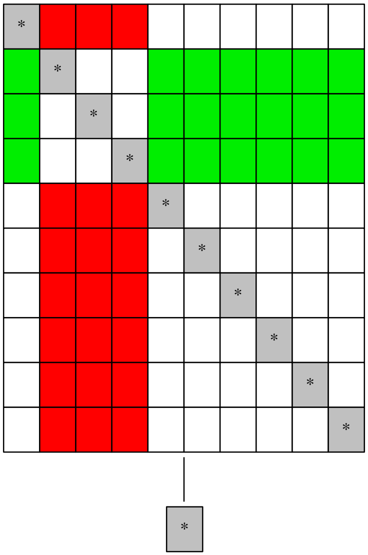

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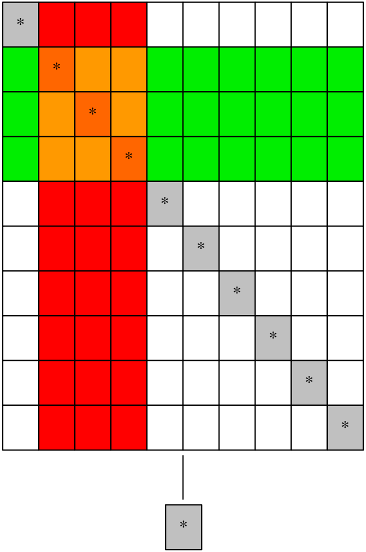

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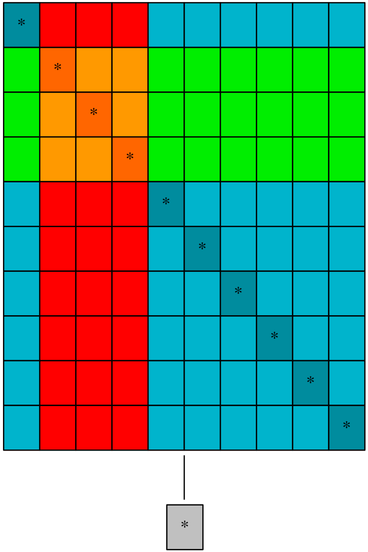

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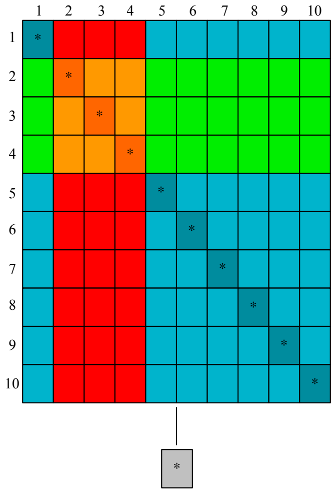

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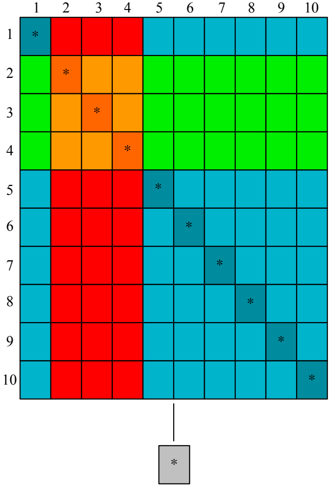

$$B(G, n) \setminus (\{2, 3, 4\} \times G \times \{1, 5, \ldots, 10\})$$

The length of the symmetric inverse monoid

To find the length of the symmetric inverse monoid it suffices to show that

$$l(B(G,n)) = n(l(G)+1) + \frac{n(n-1)}{2}|G| + (n-1).$$

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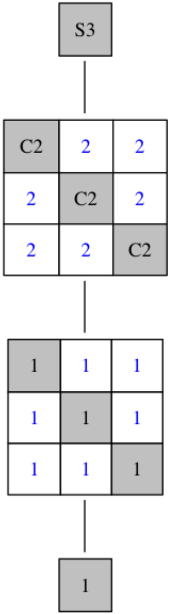
It turns out that the latter exceeds the former.

A longest chain in I_3

0



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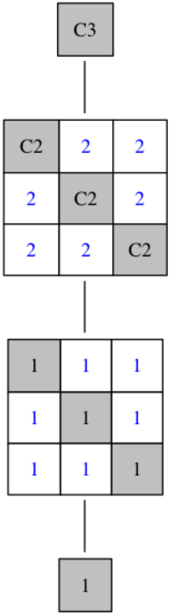
1



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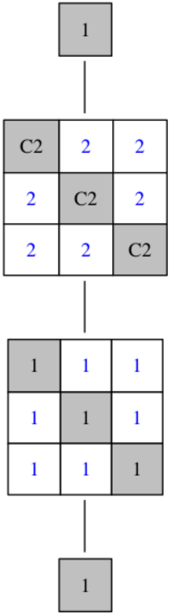
2



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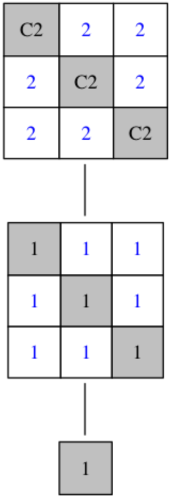

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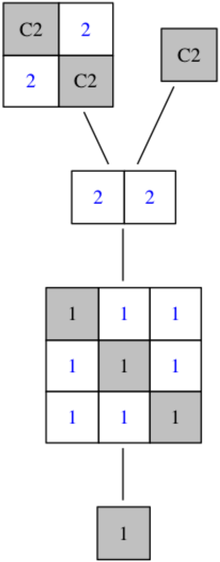

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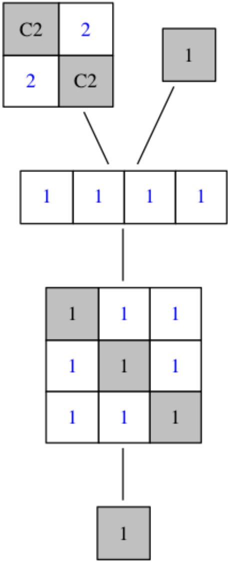

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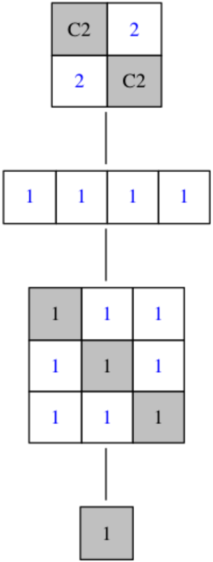

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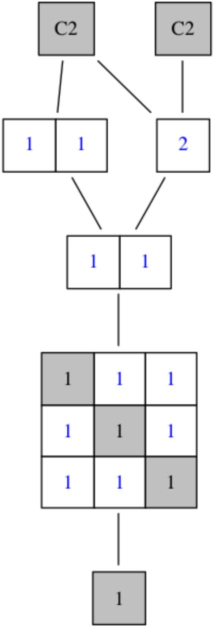

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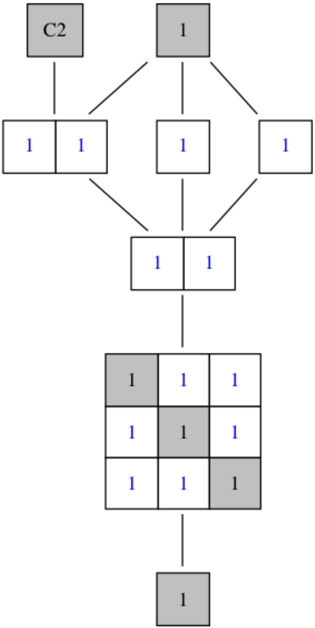
8



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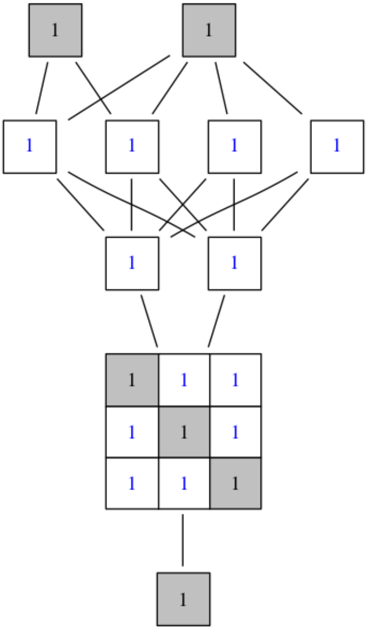

A longest chain in I_3

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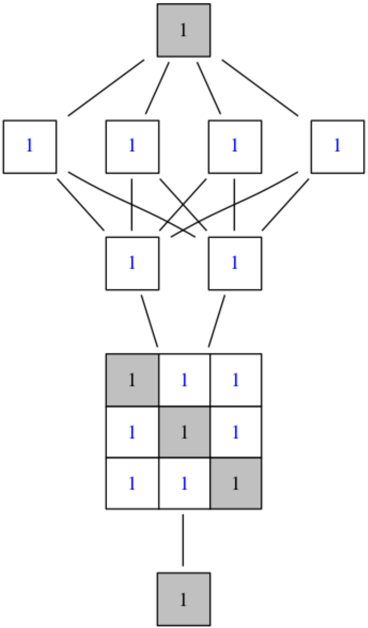
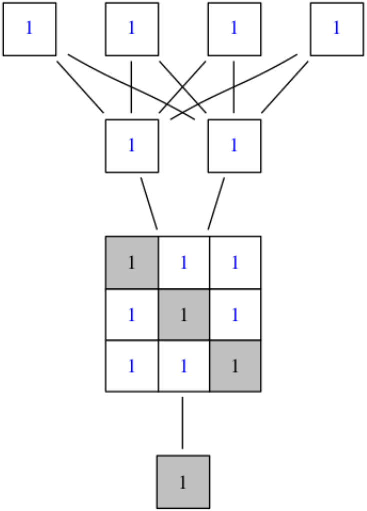


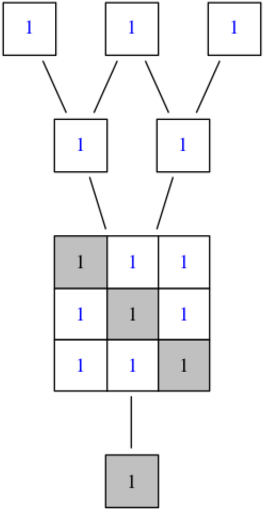
A longest chain in I_3

11



A longest chain in I_3

12



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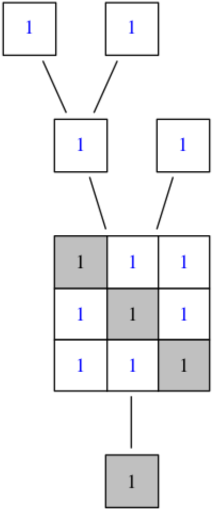
13



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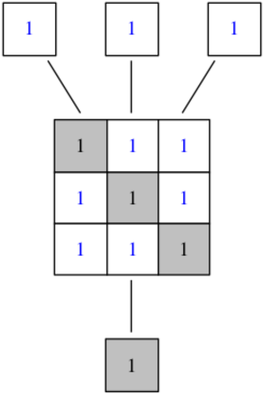

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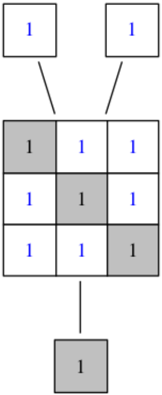
15



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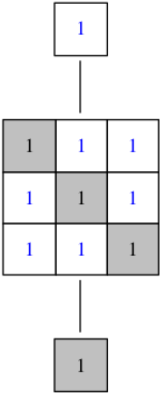
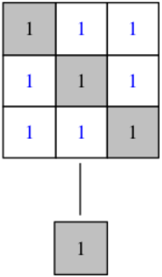


A longest chain in I_3

17



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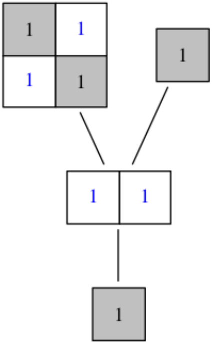
18



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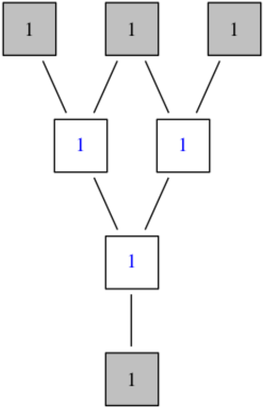
19



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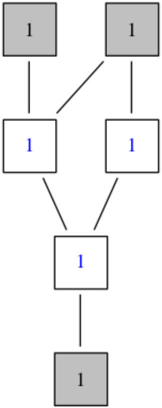
20



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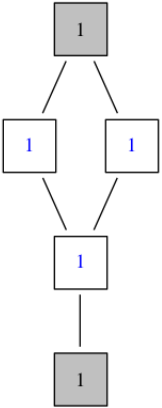
21



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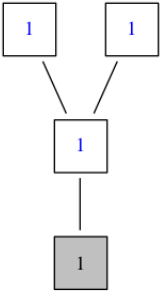
22



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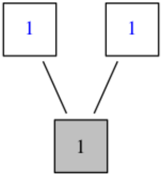
23



A longest chain in I_3

24

A longest chain in I_3

25

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If f and g are transformations with $|\operatorname{im}(f)| = |\operatorname{im}(g)| = k$, then $|\operatorname{im}(fg)| = k$ if and only if $\operatorname{im}(f)$ is a transversal of $\ker(g)$.

Problem?

A "rectangle" of zeros is: a set P_k of k-partitions of $\{1, \ldots, n\}$, and a set S_k of k-subsets, with the property that no element of S_k is a transversal for any element of P_k.

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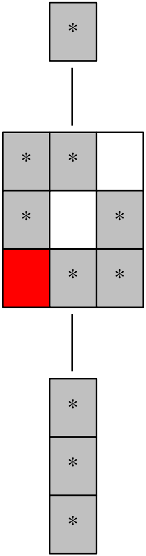
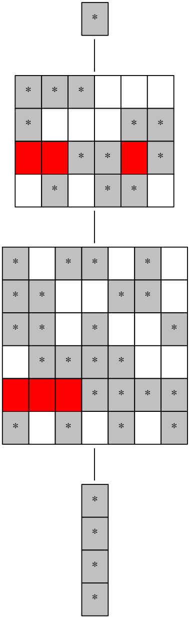
Maximise:

$$|P_k| \cdot |S_k|$$

Maximal leagues



Maximal leagues



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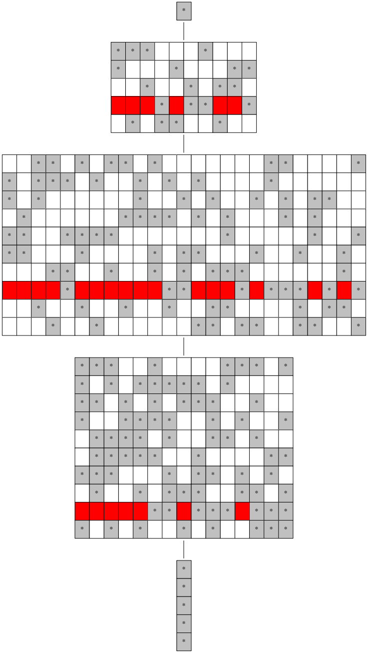

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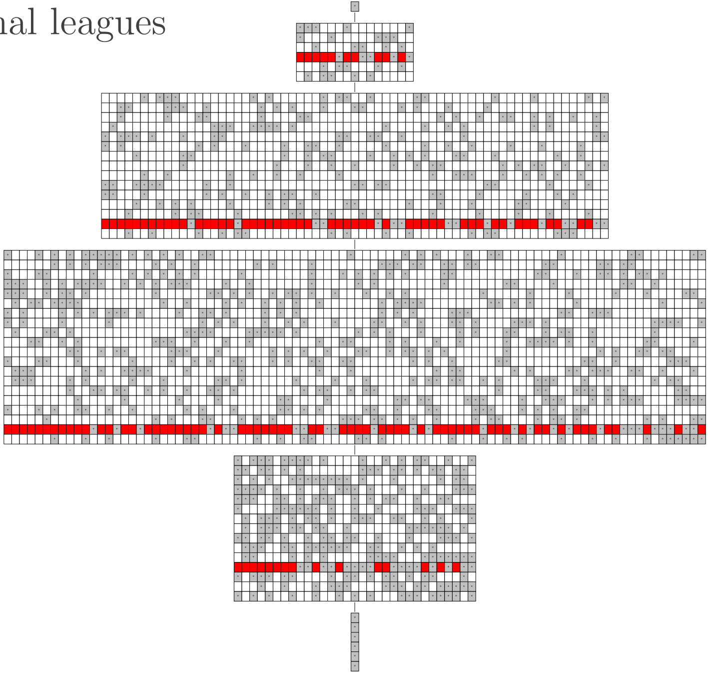

Two strategies

1. Let P_k consist of all k-partitions having n as a singleton, and let S_k consist of all k-subsets not containing n. Then (P_k, S_k) is a "rectangle" of zeros and

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$$|P_k| \cdot |S_k| = \binom{n-1}{k} S(n-1, k-1).$$

2. Let P_k be the set of all k-partitions with 1 and 2 in the same class, and let S_k be the set of all k-subsets containing 1 and 2. Then (P_k, S_k) is a "rectangle" of zeros and

$$|P_k| \cdot |S_k| = \binom{n-2}{k-2} S(n-1, k).$$

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Strategy 1 is better for large k and Strategy 2 for small k.

Some values

n	Total	$k = 2$	3	4	5	6
3	$2, 2$	$1, 1$				
4	$24, 18$	$3, 3$	$3, 2$			
5	$330, 326$	$9, 7$	$28, 28$	$6, 6$		
6	$5382, 5130$	$21, 15$	$150, 150$	$125, 125$	$12, 10$	
7	$98250, 93782$	$45, 31$	$760, 620$	$1350, 1350$	$390, 390$	$20, 15$

The left hand values are the actual maximum size of a "rectangle" of zeros as computed using GAP and Minion.

The right hand values are the maximum of the values obtained from strategies 1 and 2 on the last slide.

Thanks!

Thanks!

The pictures in this talk were produced automagically using the Semigroups package for GAP:

J. D. Mitchell et al., Semigroups - GAP package, Version 2.4.1, May, 2015; `http://tinyurl.com/semigroups`.

The algorithm for computing maximal subsemigroups of arbitrary semigroups mentioned above is also implemented in Semigroups.