Atoms and coatoms in the lattices of congruence lattices of algebras on a (finite) set

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Outline

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Compatible relations — congruences, quasiorders

$\langle A, F\rangle$ universal algebra

compatible (invariant) relation $q \subseteq A \times A$:
For each $f \in F$ (n-ary) we have $f \triangleright q$ (f preserves q), i.e.

$$(a_1, b_1), \ldots, (a_n, b_n) \in q \implies (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in q\,.$$

Con(A, F) compatible equivalence relations = *congruences*
Pord$\langle A, F\rangle$ compatible partial orders (refl., trans., antisymmetric)
Generalization of Pord$\langle A, F\rangle$ and Con$\langle A, F\rangle$:
Quord$\langle A, F\rangle$ compatible *quasiorders* (reflexive, transitive)

Remark

(Con$\langle A, F\rangle, \subseteq$) is a lattice and it is a complete sublattice of the lattice (Eq$(A), \subseteq$) of all equivalence relations on A.

Problem

Describe the lattice
$\mathcal{E} := (\{\mathrm{Con}\langle A, F\rangle \mid F$ *set of operations on* $A\}, \subseteq)$.

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Reduction to (mono)unary algebras

$H :=$ unary polynomial operations of $\langle A, F\rangle$ (i.e. $H = \langle F \cup C\rangle^{(1)}$).

It is well-known that

$$\mathrm{Con}\langle A, F\rangle = \mathrm{Con}\langle A, H\rangle$$
$$\mathrm{Con}\langle A, H\rangle = \bigcap_{f \in H} \mathrm{Con}\langle A, f\rangle.$$

Thus $\mathcal{E} = (\{\mathrm{Con}\langle A, H\rangle \mid H \leq A^A\}, \subseteq)$.

Description of $\mathcal{E}$: look for ∨- and ∧-irreducible elements,
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Notions and notations

$\vee$-irreducibles (in particular atoms) of $\mathcal{E}$

Coatoms of $\mathcal{E}$

What are the $\vee$-irreducibles?

$\varkappa \in L := \mathrm{Con}(A,H) \implies H \subseteq \mathrm{End}\,\varkappa \implies$
$\mathrm{Con}(A,H) \supseteq \mathrm{Con}(A,\mathrm{End}\,\varkappa) \implies L = \bigcup_{\varkappa \in L} \mathrm{Con}(A,\mathrm{End}\,\varkappa)$.

Thus each $\vee$-irreducible element $L = \mathrm{Con}(A,H)$ in $\mathcal{E}$ is of the form

$$L_{\varkappa} := \mathrm{Con}(A,\mathrm{End}\,\varkappa) \text{ for some } \varkappa \in \mathrm{Con}(A).$$

Question: Which $\varkappa$ yield $\vee$-irreducibles?

Answer: Every nontrivial $\varkappa$

Proof: By a result of Pöschel/Radeleczki 2007:

$$\mathrm{Quord}(A,\mathrm{End}\,\varkappa) = \{\Delta, \varkappa, \nabla\} \text{ for } \varkappa \in \mathrm{Eq}(A).$$

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$\mathcal{E}$ is atomistic

Summarizing we have (for arbitrary A):

Theorem

The completely ∨-irreducibles of $\mathcal{E}$ are exactly the congruence lattices of the form

$$L_\varkappa := \mathrm{Con}(A, \mathrm{End}\,\varkappa) = \{\Delta, \varkappa, \nabla\}$$

where $\varkappa \in \mathrm{Eq}(A) \setminus \{\Delta, \nabla\}$ is an arbitrary equivalence relation. Moreover, each ∨-irreducible is an atom in $\mathcal{E}$, i.e. the lattice $\mathcal{E}$ is atomistic.

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$\wedge$-irreducibles, in particular coatoms, are of the form $\mathrm{Con}(A, f)$ for some nontrivial $f \in A^A$.

Which f yield coatoms?

Strategy: search for candidates among the coatoms in the lattice $\mathcal{L}$ of quasiorders lattices $\mathrm{Quord}(A, f)$ (because these are known).

Why this must work?

Proposition

A lattice L is a coatom in $\mathcal{E}$ only if there exist an operation $f \in A^A$ such that $L = \mathrm{Con}(A, f)$ and $\mathrm{Quord}(A, f)$ is a coatom in $\mathcal{L}$.

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Proof ("only if" part)

to show: L coatom of $\mathcal{E} \implies L = \mathrm{Con}(A, f)$ for some f with $\mathrm{Quord}(A, f)$ coatom in $\mathcal{L}$.

Proof. Let $L = \mathrm{Con}(A, g)$ be a coatom in $\mathcal{E}$.

$\implies \mathrm{Con}(A, g) \neq \mathrm{Eq}(A) \implies \mathrm{Quord}(A, g) \neq \mathrm{Quord}(A)$

$\implies \exists$ coatom $\mathrm{Quord}(A, f)$ in $\mathcal{L}$: $\mathrm{Quord}(A, g) \subseteq \mathrm{Quord}(A, f)$

$\implies \mathrm{Con}(A, g) = \mathrm{Eq}(A) \cap \mathrm{Quord}(A, g) \subseteq \mathrm{Eq}(A) \cap \mathrm{Quord}(A, f) = \mathrm{Con}(A, f)$

$\mathrm{Con}(A, g)$ is a coatom

$\implies \mathrm{Con}(A, g) = \mathrm{Con}(A, f)$ (q.e.d.)

or $\mathrm{Con}(A, f) = \mathrm{Eq}(A)$

the latter is impossible since it would give

$$\{\mathrm{id}_A\} = \mathrm{End}\,\mathrm{Eq}(A) = \mathrm{End}\,\mathrm{Con}(A, f) \supseteq \mathrm{End}\,\mathrm{Quord}(A, f)$$

$\implies \mathrm{Quord}(A, f) = \mathrm{Quord}\,\mathrm{End}\,\mathrm{Quord}(A, f) = \mathrm{Quord}\{\mathrm{id}_A\} = \mathrm{Quord}(A)$,

a contradiction!

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$$\implies \exists \text{ coatom } \mathrm{Quord}(A,f) \text{ in } \mathcal{L} : \mathrm{Quord}(A,g) \subseteq \mathrm{Quord}(A,f)$$

$$\implies \mathrm{Con}(A,g) = \mathrm{Eq}(A) \cap \mathrm{Quord}(A,g) \subseteq \mathrm{Eq}(A) \cap \mathrm{Quord}(A,f) = \mathrm{Con}(A,f)$$

$\mathrm{Con}(A,g)$ is a coatom

$$\implies \mathrm{Con}(A,g) = \mathrm{Con}(A,f) \text{ (q.e.d.)}$$

or $\mathrm{Con}(A,f) = \mathrm{Eq}(A)$

the latter is impossible since it would give

$$\{\mathrm{id}_A\} = \mathrm{End}\,\mathrm{Eq}(A) = \mathrm{End}\,\mathrm{Con}(A,f) \supseteq \mathrm{End}\,\mathrm{Quord}(A,f)$$

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a contradiction!

Proof ("only if" part)

to show: L coatom of $\mathcal{E}$ $\implies$ $L = \mathrm{Con}(A, f)$ for some f with $\mathrm{Quord}(A, f)$ coatom in $\mathcal{L}$.

Proof. Let $L = \mathrm{Con}(A, g)$ be a coatom in $\mathcal{E}$.

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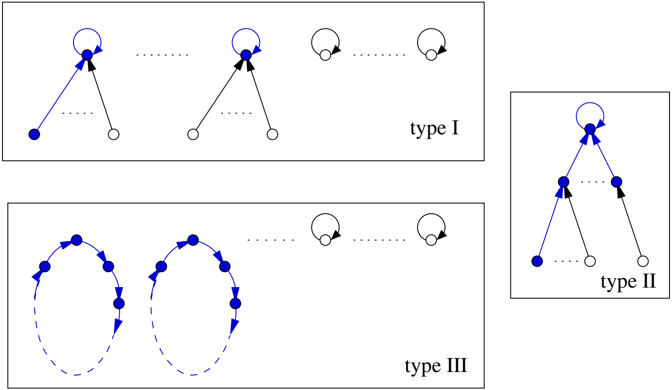
a contradiction!

The coatoms of $\mathcal{L}$

[JPR 2015] (Algebra Universalis, to appear): The coatoms of $\mathcal{L}$ are of the form $\mathrm{Quord}(A, f)$ where $f : A \to A$ is of one of the following types:



The coatoms of $\mathcal{E}$

Theorem

The coatoms of $\mathcal{E}$ are exactly the congruence lattices of the form $\mathrm{Con}(A, f)$ *where $f \in A^A$ is nontrivial (i.e., not constant and not the identity) and satisfies*

(I) $f^2 = f$, *or*

(II) f^2 *is a constant, say* 0, *and* $|[0]_{\ker f}| \geq 3$, *or*

(III) $f^p = \mathrm{id}_A$ *for some prime p such that the permutation f has at least two cycles of length p.*

References

D. Jakubíková-Studenovská, R. Pöschel, and S. Radeleczki, *The lattice of quasiorder lattices of algebras on a finite set*. Algebra Universalis , to appear.

R. Pöschel and S. Radeleczki, *Endomorphisms of quasiorders and related lattices*. In: G. Dorfer, G. Eigenthaler, H. Kautschitsch, W. More, and W.B. Müller (Eds.), *Contributions to General Algebra 18*, Verlag Heyn GmbH & Co KG, 2008, pp. 113–128, (Proceedings of the Klagenfurt Conference 2007 (AAA73+CYA22), Febr. 2007).


Thank you for your ATTENTION!