Admissible Rules of (Fragments of) R-Mingle

Laura Janina Schnüriger

joint work with George Metcalfe
Universität Bern

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What and why?

Motivation

Shorten and fasten proofs by adding “short-cuts”.

But we don’t want to have new theorems.

admissible

Such a rule is called *admissible*.

R-Mingle

RM	Relevance logic R with Mingle
Mingle	$p \to (p \to p)$
RM^t	RM with additional constant t
Language	$\mathcal{L}_t = \{\wedge, \vee, \to, \cdot, \neg, t\}$

Definition

- rules are denoted by Γ/φ for finite $\Gamma \cup \{\varphi\} \subset \mathrm{Fm}_{\mathcal{L}}$
- Γ/φ is derivable in a logic L if $\Gamma \vdash_{\mathrm{L}} \varphi$
- Γ/φ is admissible in a logic L if for all substitutions (homomorphisms) $\sigma\colon \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}$:

$$\vdash_{\mathrm{L}} \sigma(\psi) \text{ for all } \psi \in \Gamma \quad \Rightarrow \quad \vdash_{\mathrm{L}} \sigma(\varphi)$$

- $\{\Gamma/\varphi \mid \Gamma/\varphi \text{ is admissible in } \mathrm{L}\} =: \mid\!\sim_{\mathrm{L}}$
- Let $\mathcal{R}$ be a set of rules.

$$\mathrm{L} + \mathcal{R} = \text{ smallest logic containing } \mathrm{L} \cup \mathcal{R}$$

- $\mathcal{R}$ is a basis for the admissible rules of L if $\mathrm{L} + \mathcal{R} = \mid\!\sim_{\mathrm{L}}$

Corresponding algebraic semantics

$$\mathbf{Z}^\circ = \langle \mathbb{Z} \setminus \{0\}, \min, \max, \to, \cdot, -, 1 \rangle$$

$$\to \quad x \to y := \begin{cases} \max\{-x, y\} & \text{if } x \leq y \\ \min\{-x, y\} & \text{if } x > y \end{cases}$$

$$\cdot \quad x \cdot y := \begin{cases} \min\{x, y\} & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |x| > |y| \end{cases}$$

$$\mathbf{Z}_{2n} = \langle \{-n, \ldots, -1, 1, \ldots, n\}, \min, \max, \to, \cdot, -, 1 \rangle$$

$$\mathbf{Z}_{2n+1} = \langle \{-n, \ldots, -1, 0, 1, \ldots, n\}, \min, \max, \to, \cdot, -, 1 \rangle$$

Sugihara Monoids

$\mathcal{SM} =$ $\mathbb{V}(\mathbf{Z}^\circ)$ the variety of Sugihara Monoids generated by $\mathbf{Z}^\circ$.

$\mathcal{SM}$ provides an equivalent algebraic semantics for RM^t

$$\{\psi \approx |\psi| \mid \psi \in \Gamma\} \vDash_{\mathcal{SM}} \varphi \approx |\varphi| \quad \Leftrightarrow: \quad \Gamma \vDash_{\mathcal{SM}} \varphi$$
$$\Leftrightarrow \quad \Gamma \vdash_{\mathrm{RM^t}} \varphi$$

for any rule Γ/φ.

This talk

Bases for admissible rules of the fragments of RM^{t} with the following languages

$\mathcal{L}_1 =$ $\{\to, t\}$

$\mathcal{L}_2 =$ $\{\to, \cdot, t\}$

$\mathcal{L}_m =$ $\{\to, \neg, t\} = \{\to, \cdot, \neg, t\}$ multiplicative fragment.

$\mathcal{SM} \upharpoonright \mathcal{L}_i$ algebraic semantics corresponding to the $\mathcal{L}_i$-fragment of RM^t , $i \in \{1, 2, m\}$

Remark *Raftery, Olson* $\mathrm{RM}^{\mathrm{t}} \upharpoonright \{\wedge, \to, \mathrm{t}\}$ has empty basis (= it is structurally complete).

Idea of how to find the bases

Recall $\mathcal{SM} = \mathbb{V}(\mathcal{SM}) = \mathbb{V}(\mathbf{Z}^\circ)$

Lemma *S.* $\mathbb{V}(\mathcal{SM} \restriction \mathcal{L}_i) = \mathbb{V}(\mathbf{Z}_4 \restriction \mathcal{L}_i), \quad i \in \{1, 2, m\}$

Idea of how to find the bases

- Recall that for two varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ we have: $\mathbb{V}_1 = \mathbb{V}_2$ iff $(\vdash_{\mathbb{V}_1} \varphi \;\Leftrightarrow\; \vdash_{\mathbb{V}_2} \varphi$ for all formulas $\varphi)$.
- A rule is admissible in $\mathrm{RM}^{\mathrm{t}} \restriction \mathcal{L}_{\mathrm{i}}$
  $\Leftrightarrow$ it is admissible in $\mathcal{SM} \restriction \mathcal{L}_i$
  $\Leftrightarrow$ it is admissible in $\mathbf{Z}_4 \restriction \mathcal{L}_i$
- Interested in algebras s.t. admissibility in $\mathbf{Z}_4 \restriction \mathcal{L}_i$ corresponds to validity in these algebras.
- Then: Axiomatize the quasivarieties generated by these algebras to get an axiomatization of the admissible rules of our fragments.

Finding the bases

Theorem Let $\mathbf{B}$ be an algebra and $\mathbf{F_B}(\omega)$ the free algebra of $\mathbb{V}(\mathbf{B})$ on countably infinite many generators. Then

$$\Gamma/\varphi \text{ is } \mathbf{B}\text{-admissible} \quad \Leftrightarrow \quad \Gamma \models_{\mathbf{F_B}(\omega)} \varphi.$$

Finding the bases

Lemma The following are equivalent:

$$(i) \quad \Gamma/\varphi \text{ is } \mathbf{B}\text{-admissible} \quad \Leftrightarrow \quad \Gamma \vDash_{\mathbf{A}} \varphi$$

$$(ii) \quad \mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\mathbf{F_B}(\omega))$$

So we want to find **A** which is "easy" to axiomatize - but how?

Lemma $\mathbf{A} \subseteq \mathbf{F_B}(\omega), \mathbf{B} \in \mathbb{H}(\mathbf{A}) \quad \Rightarrow \quad \mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\mathbf{F_B}(\omega))$

The algebras in our case

Lemma S. Let $\mathbf{Z}_4' \subset (\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_1$, $\mathbf{Z}_4'' \subset (\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_2$, $(\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_m$ be the algebras pictured. Then

$$
\begin{aligned}
(i) \quad & \mathbb{Q}(\mathbf{F}_{\mathbf{Z}_4 \restriction \mathcal{L}_1}(\omega)) = \mathbb{Q}(\mathbf{Z}_4') \\
(ii) \quad & \mathbb{Q}(\mathbf{F}_{\mathbf{Z}_4 \restriction \mathcal{L}_2}(\omega)) = \mathbb{Q}(\mathbf{Z}_4'') \\
(iii) \quad & \mathbb{Q}(\mathbf{F}_{\mathbf{Z}_4 \restriction \mathcal{L}_m}(\omega)) = \mathbb{Q}((\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_m)
\end{aligned}
$$

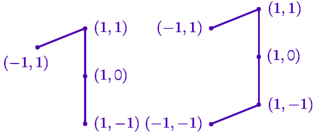

Figure: $\mathbf{Z}_4'$ and $\mathbf{Z}_4''$

Definition

- $|\psi| := \psi \to \psi$
- $\varphi \Rightarrow \psi := (\varphi \to |\psi|) \to (\varphi \to \psi)$
- $\{p, p \Rightarrow q\}/q$ $\quad (A)$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \cdot (\psi \to \varphi)$
- $\{\neg(|p_1| \leftrightarrow \ldots \leftrightarrow |p_n|)\}/q$ $\quad (R_n), \quad n \in \mathbb{N}$.

The Bases

Lemma S. We have the following axiomatizations:

(*i*) $\mathrm{RM}^{\mathrm{t}} \restriction \mathcal{L}_1 + (A)$ has equivalent q.v. $\mathbb{Q}(\mathbf{Z}_4')$
(*ii*) $\mathrm{RM}^{\mathrm{t}} \restriction \mathcal{L}_2 + (A)$ has equivalent q.v. $\mathbb{Q}(\mathbf{Z}_4'')$
(*iii*) $\mathrm{RM}^{\mathrm{t}} \restriction \mathcal{L}_{\mathrm{m}} + (A) + \{(R_n)\}_{n\in\mathbb{N}}$ has eq. q.v. $\mathbb{Q}((\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_m)$

Theorem S. Then as a Corollary of this lemma

(*i*) $\{(A)\}$ is a basis for the $\{\rightarrow, t\}$- and $\{\rightarrow, \cdot, t\}$-fragment of RM^{t}.
(*ii*) $\{(A)\} \cup \{(R_n)\}_{n\in\mathbb{N}}$ is a basis for $\mathrm{RM}^{\mathrm{t}} \restriction \{\rightarrow, \neg, \mathrm{t}$

References

- G. Metcalfe and C. Röthlisberger. *Admissibility in finitely generated quasivarieties.* Logical Methods in Computer Science, vol. 9 (2013), no. 2, pp. 119.
- C. Röthlisberger. *Admissibility in finitely generated quasivarieties.* PhD thesis, 2013.
- G. Metcalfe. *An Avron rule for fragments of R-mingle.* Journal of Logic and Computation, to appear.
- J. Olson and J. Raftery. *Positive Sugihara monoids.* Algebra Universalis, vol. 57 (2005), pp. 7599.