TECHNISCHE UNIVERSITÄT WIEN

CSP dichotomy for solving systems of equations on 3

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CSP jargon

Constraint language

…a finite set of relations Γ on a finite domain

CSP template

…a relational structure $\underset{\sim}{\mathbf{T}} = \langle T; \Gamma \rangle$ … Γ constraint language

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Let $\langle T; \Gamma \rangle$ be a CSP template, V a finite set (of variables)

Constraint over Γ and V

...a pair $(R, (v_1, \ldots, v_n))$ s.t. $R \in \Gamma$, n-ary, $(v_1, \ldots, v_n) \in V^n$

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Satisfaction relation (assignment $s \in V^T$)

$s\colon V \longrightarrow T \models (R, (v_1, \dots, v_n)) \iff s \circ (v_1, \dots, v_n) \in R.$

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CSP(Γ) ... a decision problem

Instance: finite set C of constraints over Γ and variables V

Question: $\exists s \colon V \longrightarrow T \; \forall c \in C \colon s \models c$?

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Instance: a conjunctive formula φ over Γ and variables V

Question: is φ satisfiable, i.e. $\exists s \colon V \longrightarrow T \colon s \models \varphi$?

CSP(Γ) ... still differently

Instance: a finite relational structure $\underset{\sim}{\mathbf{V}} = \left\langle V; \left(\tilde{R}\right)_{R \in \Gamma} \right\rangle$

Question: $\exists s \colon V \longrightarrow T \colon s$ hom from $\underset{\sim}{\mathbf{V}}$ to $\underset{\sim}{\mathbf{T}}$?

pp-CSP

Γ ... constraint language over domain T

pp-CSP(Γ) ... a more involved decision problem

Instance: a primitive positive formula φ over $\Gamma \cup \{\Delta\}$ and variables V

Question: is φ satisfiable, i.e. $\exists s \colon V \longrightarrow T \colon s \models \varphi$?

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Lemma (Jeavons, 1998)

For constraint languages Γ: pp-CSP(Γ) $\leq_{\mathrm{m}}$ CSP(Γ)

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Lemma (Jeavons, 1998)

For constraint languages Γ: $\text{pp-CSP}(\Gamma) \leq_{\text{m}} \text{CSP}(\Gamma)$

Corollary (Jeavons, 1998)

For constraint languages Γ,Γ':
$\Gamma' \subseteq [\Gamma]_{\text{R}_T} \implies \text{CSP}(\Gamma') \leq_{\text{m}} \text{CSP}(\Gamma)$

CSP dichotomy conjecture

Fix a CSP template $\underset{\sim}{\mathbf{T}}$.

Time complexity

CSP $(\underset{\sim}{\mathbf{T}})$ is always in $\mathtt{NP}$, some CSP's are in $\mathtt{P}$.

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Schaefer, 1978: $|T| = 2 \Rightarrow$ CSP $(\underset{\sim}{\mathbf{T}})$ in P xor NP-complete.
Conjecture: This extends to all finite domains.

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CSP dichotomy conjecture holds for...

$|T| = 2$ Schaefer, 1978

$|T| = 3$ Bulatov, 2006

$|T| = 4$ Marković, 2012

Equational CSP

T be a finite set, $F \subseteq \mathrm{O}_T$ finite, V finite set (of variables).

Equational constraint over F and V...

an equality $s \doteq t$ where s, t are terms over F and V

Satisfaction

assignment $m\colon V \longrightarrow T \models s \doteq t$ iff $[\![s]\!]_{m,F} = [\![t]\!]_{m,F}$

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EqCSP(F) ...a decision problem

Instance: finite set C of equational constraints over Γ and variables V

Question: $\exists m\colon V \longrightarrow T \; \forall c \in C\colon m \models c$?

Equational CSP is nothing new

Let $F \subseteq O_T$ be finite on a finite domain T.

Lemma

EqCSP $F \leq_m$ pp-CSP $F^\bullet \leq_m$ CSP $F^\bullet$.

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Lemma

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For instance...

- $f(g_1(x,y), g_2(y,z)) \doteq h(x,y,z,z)$
- $\exists uv\colon u = f(g_1(x,y), g_2(y,z)) \wedge v = h(x,y,z,z) \wedge \Delta(u,v)$
- $\exists u u_1 u_2 v\colon \; u = f(u_1,u_2) \wedge v = h(x,y,z,z) \wedge u_1 = g_1(x,y) \wedge u_2 = g_2(y,z) \wedge \Delta(u,v)$
- $\exists u u_1 u_2 v\colon \; f^\bullet(u_1,u_2,u) \wedge h^\bullet(x,y,z,z,v) \wedge g_1^\bullet(x,y,u_1) \wedge g_2^\bullet(y,z,u_2) \wedge \Delta(u,v)$

CSP dichotomy revisited

To prove CSP dichotomy for finite domains...

Feder/Madelaine/Stewart (2003) | Broniek (2008):
...it suffices to establish the dichotomy conjecture for
$\underset{\sim}{\mathbf{T}} = \langle T; \Gamma \rangle$, where $\Gamma = \{f^\bullet, g^\bullet\}$ for some $f, g \in \mathrm{O}_T^{(1)}$.

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Connection to centraliser clones

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Fun fact (Burris/Willard, 1987)

There are only finitely many centraliser clones on finite sets.

From now on $|T| = 3$!!!

Goal

- Although Bulatov solved $|T| = 3\dots$

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- Although Bulatov solved $|T| = 3\ldots$
- $F^\bullet$-templates = special case, so classification easier(?)
- $|T| = 3$ as a test case for larger domains
- application of new/refined methods (cp. pre 2006)
- precise description of the border line possible

Centraliser clones

$|T| = 2$, А. В. Кузнецов (Kuznecov), 1977

description of all 25 clones

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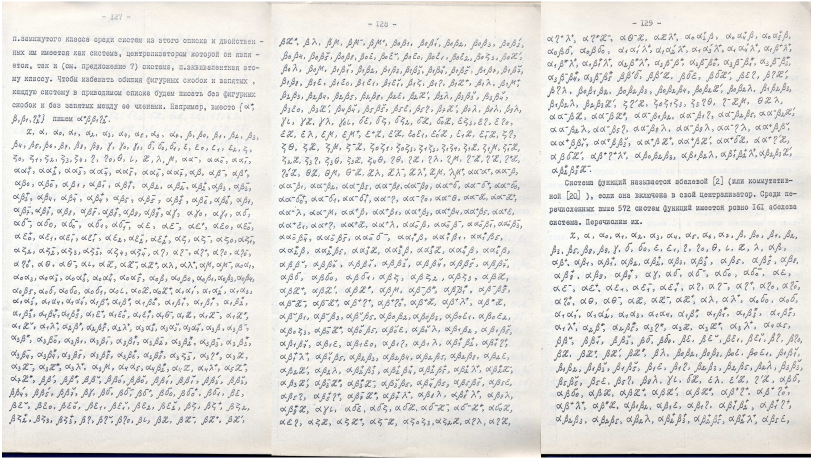
description of all 25 clones

$|T| = 3$, А. Ф. Данильченко (Danil'čenko), 1974–79

- in a 141 pages thesis & 2–3 cryptic papers
- determined all $\bigcap$-irreducible clones in the lattice
- described all 2986 centraliser clones
- 572 up to inner symmetries...
- ...with generating systems w.r.t. pp-definability

(One of) Даниьченко's theorem(s)



In LaTeX it fits on one page, actually

$x, \alpha, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \gamma, \gamma_0, \gamma_1, \delta, \delta_0, \delta_1,$
$\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2, \zeta, \zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \eta, \eta_0, \theta, \iota, \varkappa, \lambda, \mu, \alpha\alpha^-, \alpha\alpha_0^-, \alpha\alpha_1^-, \alpha\alpha_1^+, \alpha\alpha_2^+, \alpha\alpha_3^-, \alpha\alpha_4^-,$
$\alpha\alpha_5^-, \alpha\alpha_6^-, \alpha\alpha_7^-, \alpha\beta, \alpha\beta^-, \alpha\beta^+, \alpha\beta_0, \alpha\beta_0^-, \alpha\beta_1, \alpha\beta_1^-, \alpha\beta_1^+, \alpha\beta_2, \alpha\beta_2^-, \alpha\beta_2^+, \alpha\beta_3, \alpha\beta_3^-, \alpha\beta_3^+,$
$\alpha\beta_4, \alpha\beta_4^-, \alpha\beta_4^+, \alpha\beta_5, \alpha\beta_5^-, \alpha\beta_5^+, \alpha\beta_6^-, \alpha\beta_6^+, \alpha\beta_7, \alpha\beta_7^-, \alpha\beta_7^+, \alpha\beta_8, \alpha\beta_8^-, \alpha\beta_8^+, \alpha\beta_9, \alpha\beta_9^+, \alpha\gamma,$
$\alpha\gamma_0, \alpha\gamma_1, \alpha\delta, \alpha\delta^-, \alpha\delta_0, \alpha\delta_0^-, \alpha\delta_1, \alpha\delta_1^-, \alpha\varepsilon, \alpha\varepsilon^-, \alpha\varepsilon^+, \alpha\varepsilon_0, \alpha\varepsilon_0^-, \alpha\varepsilon_0^+, \alpha\varepsilon_1, \alpha\varepsilon_1^-, \alpha\varepsilon_1^+, \alpha\varepsilon_2,$
$\alpha\varepsilon_2^-, \alpha\varepsilon_2^+, \alpha\zeta, \alpha\zeta^-, \alpha\zeta_0, \alpha\zeta_1^-, \alpha\zeta_2, \alpha\zeta_2^-, \alpha\zeta_3, \alpha\zeta_3^-, \alpha\zeta_4, \alpha\zeta_4^-, \alpha\eta, \alpha\eta^-, \alpha\eta^+, \alpha\eta_0, \alpha\eta_0^-, \alpha\eta_0^+,$
$\alpha\theta, \alpha\theta^-, \alpha\iota, \alpha\varkappa, \alpha\varkappa^-, \alpha\varkappa^+, \alpha\lambda, \alpha\lambda^+, \alpha\mu, \alpha\mu^-, \alpha_0\alpha_1, \alpha_0\alpha_3, \alpha_0\alpha_3^-, \alpha_0\alpha_3^+, \alpha_0\alpha_4^+, \alpha_0\alpha_5^+, \alpha_0\beta,$
$\alpha_0\beta_0, \alpha_0\beta_1, \alpha_0\beta_3, \alpha_0\beta_4, \alpha_0\beta_5, \alpha_0\delta, \alpha_0\delta_0, \alpha_0\delta_1, \alpha_0\iota, \alpha_0\varkappa, [\alpha_0\lambda^+]^{\ddagger}, \alpha_1\alpha_1', \alpha_1\alpha_2', \alpha_1\alpha_3, \alpha_1\alpha_3',$
$\alpha_1\alpha_4, \alpha_1\alpha_4', \alpha_1\beta^+, \alpha_1\beta^*, \alpha_1\beta_0^*, \alpha_1\beta_1^+, \alpha_1\beta_1^*, \alpha_1\beta_2^*, \alpha_1\beta_3^*, \alpha_1\beta_4^*, \alpha_1\beta_5^*, \alpha_1\varepsilon^+, \alpha_1\varepsilon_0^+, \alpha_1\varepsilon_1^*, \alpha_1\theta^-,$
$\alpha_1\varkappa, \alpha_1\varkappa^-, \alpha_1\varkappa^*, \alpha_1\varkappa^\cup, \alpha_1\lambda^+, \alpha_2\beta^*, \alpha_2\beta_5^*, \alpha_2\lambda^+, \alpha_3\alpha_3^*, \alpha_3\alpha_3^\cup, \alpha_3\alpha_4^\cup, \alpha_3\beta, \alpha_3\beta^-, \alpha_3\beta^*, \alpha_3\beta_0^-,$
$\alpha_3\beta_1, \alpha_3\beta_1^-, \alpha_3\beta_1^*, \alpha_3\beta_2^-, \alpha_3\beta_2^*, \alpha_3\beta_3^-, \alpha_3\beta_3^*, \alpha_3\beta_4^-, \alpha_3\beta_4^*, \alpha_3\beta_5^-, \alpha_3\beta_5^*, \alpha_3\beta_6^*, \alpha_3\beta_8^*, \alpha_3\zeta_3^-,$
$\alpha_3\eta^*, \alpha_3\varkappa, \alpha_3\varkappa^-, \alpha_3\varkappa^*, \alpha_3\lambda^+, \alpha_3\mu, \alpha_4\alpha_5, \alpha_4\beta_2^*, \alpha_4\varkappa, \alpha_4\lambda^+, \alpha_5\varkappa^*, \alpha_7\varkappa^*, \beta\beta', \beta\beta^*, \beta\beta^\cup, \beta\beta_0',$
$\beta\beta_0^\cup, \beta\beta_1', \beta\beta_1^\cup, \beta\beta_3', \beta\beta_3^\cup, \beta\beta_4', \beta\beta_5', \beta\beta_7^\cup, \beta\gamma, \beta\delta, \beta\delta^-, \beta\delta^+, \beta\delta_0, \beta\delta_0^+, \beta\delta_1, \beta\varepsilon, \beta\varepsilon^\cup, \beta\varepsilon_0,$
$\beta\varepsilon_0^\cup, \beta\varepsilon_1, \beta\varepsilon_1^\cup, \beta\varepsilon_2, \beta\varepsilon_2^\cup, \beta\zeta, \beta\zeta^+, \beta\zeta_2, \beta\zeta_2^+, \beta\zeta_3, \beta\zeta_3^+, \beta\eta, \beta\eta^\cup, \beta\eta_0, \beta\iota, \beta\varkappa, \beta\varkappa^-, \beta\varkappa^+, \beta\not\varkappa,$
$\beta\varkappa^*, \beta\lambda, \beta\mu, \beta\mu^-, \beta\mu^+, \beta_0\beta_1, \beta_0\beta_1', \beta_0\beta_2, \beta_0\beta_3, \beta_0\beta_3', \beta_0\beta_4, \beta_0\beta_5^\cup, \beta_0\beta_8, \beta_0\varepsilon, \beta_0\varepsilon^\cup, \beta_0\varepsilon_0, \beta_0\varepsilon_1,$
$\beta_0\varepsilon_2, \beta_0\zeta_3, \beta_0\not\varkappa, \beta_0\lambda, \beta_0\mu, \beta_1\beta_1^\cup, \beta_1\beta_2, \beta_1\beta_3, \beta_1\beta_3^\cup, \beta_1\beta_4^\cup, \beta_1\beta_5^\cup, \beta_1\beta_7, \beta_1\beta_7^\cup, \beta_1\beta_8, \beta_1\varepsilon, \beta_1\varepsilon_0,$
$\beta_1\varepsilon_1, \beta_1\varepsilon_1^\cup, \beta_1\zeta, \beta_1\eta, \beta_1\varkappa^+, \beta_1\lambda, \beta_1\mu^+, \beta_2\beta_3, \beta_2\beta_4, \beta_2\beta_5, \beta_2\beta_8, \beta_2\varepsilon, \beta_2\not\varkappa, \beta_2\lambda, \beta_3\beta_3^\cup, \beta_3\beta_4^\cup,$
$\beta_3\varepsilon_0, \beta_3\not\varkappa, \beta_4\beta_4^\cup, \beta_5\beta_5^\cup, \beta_5\varepsilon, \beta_5\eta, \beta_7\not\varkappa, \beta_7\lambda, \beta_8\lambda, \beta_9\lambda, \gamma\iota, \gamma\varkappa, \gamma\lambda, \gamma_0\iota, \delta\varepsilon, \delta\zeta, \delta\zeta_2, \delta\varkappa, \delta_0\varkappa,$
$\varepsilon\zeta_3, \varepsilon\eta, \varepsilon\eta_0, \varepsilon\varkappa, \varepsilon\lambda, \varepsilon\mu, \varepsilon\mu^+, \varepsilon^+\varkappa, \varepsilon'\varkappa, \varepsilon_0\varepsilon_1, \varepsilon_0'\varkappa, \varepsilon_1\varkappa, \varepsilon_1^-\varkappa, \zeta\eta, \zeta\theta, \zeta\varkappa, \zeta\mu, \zeta^-\varkappa, \zeta_0\zeta_1, \zeta_0\zeta_3,$
$\zeta_1\zeta_3, \zeta_1\zeta_4, \zeta_1\varkappa, \zeta_1\mu, \zeta_1^-\varkappa, \zeta_2\varkappa, \zeta_3\eta, \zeta_3\theta, \zeta_3\varkappa, \zeta_4\theta, \eta\theta, \eta\varkappa, \eta\lambda, \eta\mu, \eta^-\varkappa, \eta'\varkappa, \eta^*\varkappa, \eta_0'\varkappa, \theta\varkappa,$
$\theta\mu, \theta^-\varkappa, \varkappa\lambda, \varkappa\lambda^-, \varkappa\lambda^+, \varkappa\mu, \lambda\mu^+, \alpha\alpha^-\alpha^+, \alpha\alpha^-\beta, \alpha\alpha^-\beta_1, \alpha\alpha^-\beta_2, \alpha\alpha^-\beta_5, \alpha\alpha^-\beta_8, \alpha\alpha^-\beta_9,$
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$\alpha\alpha_0^-\beta^-, \alpha\alpha_0^-\beta_1^-, \alpha\alpha_0^-\beta_3^-, \alpha\alpha_0^-\beta_4^-, \alpha\alpha_0^-\beta_5^-, \alpha\alpha_0^-\delta^-, \alpha\alpha_1^*\beta, \alpha\alpha_1^*\beta_1, \alpha\alpha_1^*\beta_5, \alpha\alpha_2^*\beta, \alpha\alpha_2^*\beta_5, \alpha\alpha_3^+\varkappa,$
$\alpha\alpha_3^*\beta, \alpha\alpha_3^*\varkappa, \alpha\alpha_4^*\beta, \alpha\alpha_6^*\beta, \alpha\beta\beta^\cup, \alpha\beta\beta_0^\cup, \alpha\beta\beta_1^\cup, \alpha\beta\beta_3^\cup, \alpha\beta\beta_4^\cup, \alpha\beta\beta_5^\cup, \alpha\beta\beta_7^\cup, \alpha\beta\delta, \alpha\beta\delta_0, \alpha\beta\delta_1,$
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$\alpha\beta^+\eta^+, \alpha\beta^+\eta_0^+, \alpha\beta^+\varkappa, \alpha\beta^+\lambda^+, \alpha\beta^*\varkappa, \alpha\beta^\cup\beta_1, \alpha\beta^\cup\beta_3, \alpha\beta^\cup\beta_5, \alpha\beta_0\beta_2, \alpha\beta_0\beta_3, \alpha\beta_0\varepsilon_1, \alpha\beta_0\varepsilon_2,$
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$\alpha\beta_7^-\varkappa^*, \alpha\beta_7^+\lambda^+, \alpha\beta_8\lambda, \alpha\beta_8^+\lambda^+, \alpha\beta_9\lambda, \alpha\beta_9^*\varkappa, \alpha\gamma\iota, \alpha\delta\varepsilon, \alpha\delta\zeta, \alpha\delta\varkappa, \alpha\delta^-\varkappa^-, \alpha\delta^-\varkappa^*, \alpha\delta_0\varkappa, \alpha\varepsilon\eta,$
$\alpha\zeta\varkappa, \alpha\zeta\varkappa^+, \alpha\zeta^-\varkappa, \alpha\zeta_0\zeta_3, \alpha\zeta_2\varkappa, \alpha\eta\lambda, \alpha\eta'\varkappa, \alpha\eta^+\lambda^+, \alpha\eta^*\varkappa^-, \alpha\theta^-\varkappa, \alpha\varkappa\lambda^+, \alpha_0\alpha_3^+\beta, \alpha_0\alpha_4^+\beta,$
$\alpha_0\alpha_5^+\beta, \alpha_0\beta\delta, \alpha_0\beta\delta_0, \alpha_1\alpha_1'\lambda^+, \alpha_1\alpha_2'\lambda^+, \alpha_1\alpha_3'\lambda^+, \alpha_1\alpha_4'\lambda^+, \alpha_1\beta^+\lambda^+, \alpha_1\beta^*\lambda^+, \alpha_1\beta_1^*\lambda^+, \alpha_2\beta^*\lambda^+,$
$\alpha_3\beta^-\beta^*, \alpha_3\beta^-\beta_0^*, \alpha_3\beta^-\beta_1^*, \alpha_3\beta^-\beta_3^*, \alpha_3\beta^-\beta_4^*, \alpha_3\beta^-\beta_5^*, \beta\beta'\delta, \beta\beta'\varkappa, \beta\delta\varepsilon, \beta\delta\not\varkappa, \beta\varepsilon\eta, \beta\eta\not\varkappa,$
$\beta\eta\lambda, \beta_0\beta_1\beta_2, \beta_0\beta_2\beta_3, \beta_0\beta_2\beta_4, \beta_0\beta_2\not\varkappa, \beta_0\beta_2\lambda, \beta_1\beta_2\beta_3, \beta_1\beta_2\lambda, \beta_2\beta_3\not\varkappa, \zeta\eta'\varkappa, \zeta_0\zeta_1\zeta_3, \zeta_3\eta\theta, \eta^-\varkappa\mu,$
$\theta\varkappa\lambda, \alpha\alpha^-\beta\varkappa, \alpha\alpha^-\beta\varkappa^*, \alpha\alpha^-\beta_1\beta_2, \alpha\alpha^-\beta_1\eta, \alpha\alpha^-\beta_2\beta_5, \alpha\alpha^-\beta_2\not\varkappa, \alpha\alpha^-\beta_2\lambda, \alpha\alpha^-\beta_5\eta, \alpha\alpha^-\beta_8\lambda,$
$\alpha\alpha^-\beta_9\lambda, \alpha\alpha^-\eta\lambda, \alpha\alpha^+\beta\beta^\cup, \alpha\alpha^+\beta\beta_1^\cup, \alpha\alpha^+\beta\beta_3^\cup, \alpha\alpha^+\beta\varkappa^+, \alpha\alpha^+\beta\not\varkappa, \alpha\alpha^+\delta\varkappa, \alpha\alpha^+\eta'\varkappa, \alpha\beta\delta\not\varkappa,$
$\alpha\beta^+\eta^+\lambda^+, \alpha\beta_0\beta_2\beta_3, \alpha\beta_1\beta_2\lambda, \alpha\beta_1^+\beta_2^+\lambda^+, \alpha\beta_2\beta_3\not\varkappa, \alpha\beta_2^*\beta_3^*\varkappa^-.$

‡This term reads "$\alpha_0\varkappa^+$" in the original article [10] and on page 127 of [11]. It has been altered using a computer program designed by the translators.

§This term reads "$\alpha\beta\beta_1^*$" in the original article [10]. It has been altered using the correct expression taken from page 128 of [11].

Used tractability criteria (I)

For a constraint language Γ, CSP(Γ) is tractable, i.e. in P, if

$\mathrm{Pol}_T\,\Gamma$ contains a...

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$\implies$ k-IP (Baker/Pixley)

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Jovanović, Marković, McKenzie, Moore, 2015

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Jovanović, Marković, McKenzie, Moore, 2015

$\Longrightarrow$ Con-SD($\wedge$) variety

$\Longrightarrow$ solvable by local consistency checking

(Barto, Kozik, 2014 + Barto manuscript, 2014)

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$$m(x_1, \ldots, m_k)\ k\text{-nu} \implies m(x_2, \ldots, x_{k+1})\ k\text{-edge}$$

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Idziak, Marković, McKenzie, Valeriote, Willard, LICS 2007

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constant constant time, always answer “yes”

Used tractability criteria (III)

$\mathbf{A} := \langle T; \mathrm{Pol}_T\, \Gamma \rangle$ for a constraint language Γ.

Marković, McKenzie, 2009 (corrected post lecture)

Suppose $\mathbf{A}$ is idempotent and find
a weak nu $w \in \mathrm{Pol}_T^{(k)} \Gamma$, $k \geq 3$, and $\theta \in \mathrm{Con}\, \mathbf{A}$ such that

- $x \circ_w y := w\,(x, \ldots, x, y)$ induces a semilattice $\langle T/\theta;\ \circ_w/\theta \rangle$ with a chain order
- $\mathbf{A}/\theta = \left\langle T/\theta; \langle \circ_w/_\theta \rangle_{\mathrm{O}_{T/\theta}} \right\rangle$
- $w/\theta\,([x_1]_\theta\,, \ldots, [x_k]_\theta) = [x_1]_\theta \circ_w/_\theta \cdots \circ_w/_\theta\, [x_k]_\theta$ for $[x_1]_\theta\,, \ldots, [x_k]_\theta \in T/\theta$
- $(x \circ_w y) \circ_w y = x \circ_w y$ for $x, y \in T$
- $x \circ_w y\ \theta\ y \implies x \circ_w y = y$ for $x, y \in T$

$\implies$ $x \circ_w (x \circ_w y) = x \circ_w y$ since
$x \circ_w (x \circ_w y)\ \theta\ (x \circ_w x) \circ_w y = x \circ_w y$
if $x\ \theta\ y$, then $x \circ_w y\ \theta\ y \circ_w y = y$, so $x \circ_w y = y$

then $\mathrm{CSP}(\Gamma)$ is tractable.

Hardness conditions

Jeavons, Cohen, Gyssens, 1997

$\Gamma \subseteq \mathrm{R}_T$ constraint language with $\mathrm{Pol}_T\,\Gamma \subseteq \left\langle \mathrm{O}_T^{(1)} \right\rangle_{\mathrm{O}_T}$ and $\mathrm{Pol}_T\,\Gamma \cap \left\{ c_a^{(1)} \;\middle|\; a \in T \right\} = \emptyset$, then $\mathrm{CSP}\,(\Gamma)$ NP-complete.

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pp-interpretability of hard problems

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NP-hard Boolean CSP's

- $\mathrm{CSP}\,(\{R_{\mathrm{nae}}\})$ $\quad R_{\mathrm{nae}} = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
  $\mathrm{Pol}_2\,\{R_{\mathrm{nae}}\} = \langle\{\neg\}\rangle_{\mathrm{O}_2}$

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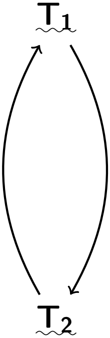
  $\mathrm{Pol}_2\{R_{\text{1-in-3}}\} = \mathrm{J}_2$

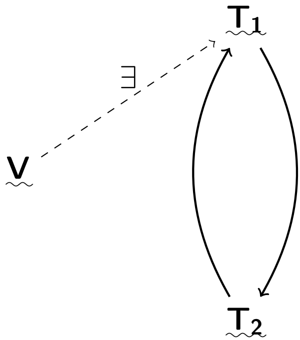
$\underset{\sim}{\mathbf{T_1}}$, $\underset{\sim}{\mathbf{T_2}}$ templates of identical type

$\underset{\sim}{\mathbf{V}}$



Retractions / idempotent endomorphisms / cores

$\underset{\sim}{\mathbf{T}_1}$, $\underset{\sim}{\mathbf{T}_2}$ templates of identical type



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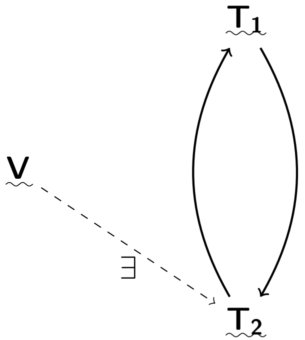
$\underset{\sim}{\mathbf{T_1}}$, $\underset{\sim}{\mathbf{T_2}}$ templates of identical type



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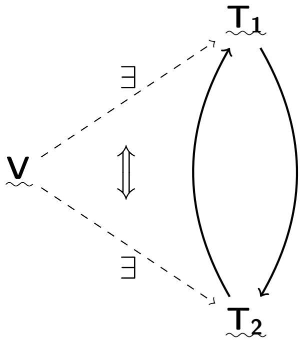

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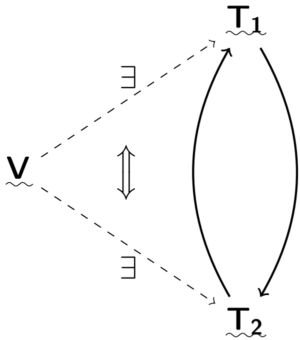
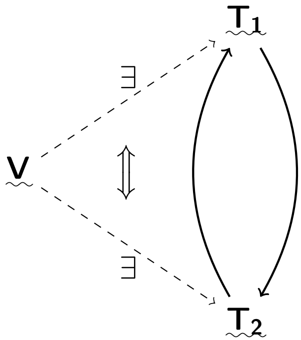


Mutual homs preserve complexity

$\underset{\sim}{\mathbf{T}}_1$, $\underset{\sim}{\mathbf{T}}_2$ templates of identical type, $\mathrm{Hom}(\underset{\sim}{\mathbf{T}}_1, \underset{\sim}{\mathbf{T}}_2) \neq \emptyset \neq \mathrm{Hom}(\underset{\sim}{\mathbf{T}}_2, \underset{\sim}{\mathbf{T}}_1)$ $\implies$ $\mathrm{CSP}(\underset{\sim}{\mathbf{T}}_1) \equiv_m \mathrm{CSP}(\underset{\sim}{\mathbf{T}}_2)$.

$\mathbf{T}_1$, $\mathbf{T}_2$ templates of identical type



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Retractions preserve complexity

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Complexity classification

Remaining possibly `NP`-complete centraliser clones:

Complexity classification

Remaining possibly NP-complete centraliser clones:

572

Complexity classification

Remaining possibly NP-complete centraliser clones:

$$572 \xrightarrow[\text{sl}]{\text{consts}} 102$$

Complexity classification

Remaining possibly NP-complete centraliser clones:

$$572 \xrightarrow[\text{sl}]{\text{consts}} 102 \xrightarrow[\text{maj}]{x-y+z} 83$$

Complexity classification

Remaining possibly NP-complete centraliser clones:

$$572 \xrightarrow[\text{sl}]{\text{consts}} 102 \xrightarrow[\text{maj}]{x-y+z} 83 \xrightarrow[\text{(only 3 needed)}]{\text{all Мальцев}} 72$$

Complexity classification

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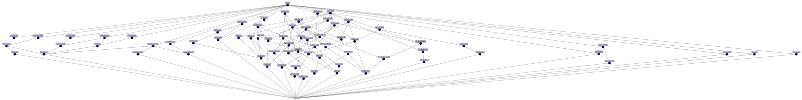
Draw the remaining poset.

72 possibly NP-complete centraliser clones



Remaining 72 rebels (out of 1 . . . 572)

49, 70, 136, 158, 179, 187, 191, 246, 256, 257, 258, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 313, 371, 374, 375, 376, 377, 379, 380, 384, 388, 391, 397, 399, 400, 401, 403, 405, 406, 407, 415, 416, 434, 441, 443, 445, 446, 447, 448, 449, 450, 451, 476, 478, 479, 484, 510, 526, 535, 536, 537, 538, 539, 550, 553, 555, 557, 558, 561, 568

Dealing with the 72 last Mohicans

NP-complete

$\{R_{\text{nae}}\} \leq_{\text{pp}} \Gamma$

261

$\{R_{\text{1-in-3}}\} \leq_{\text{pp}} \Gamma$

70, 136, 179, 377, 384, 407

$\text{Pol}_3\, \Gamma$ ess. exactly unary (Jeavons et al.)

313

Tractable

JMMM operations

400, 401, 449

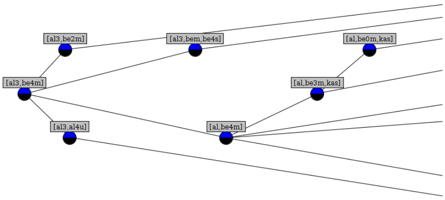
Marković/McKenzie criterion

375, 391, 561

$\{0, 1\}$-retract in P

158, 403, 406, 447

Thank you for your attention.



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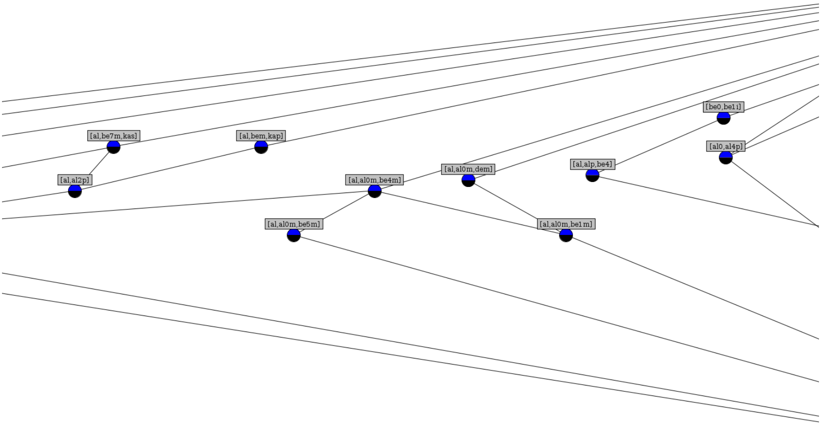
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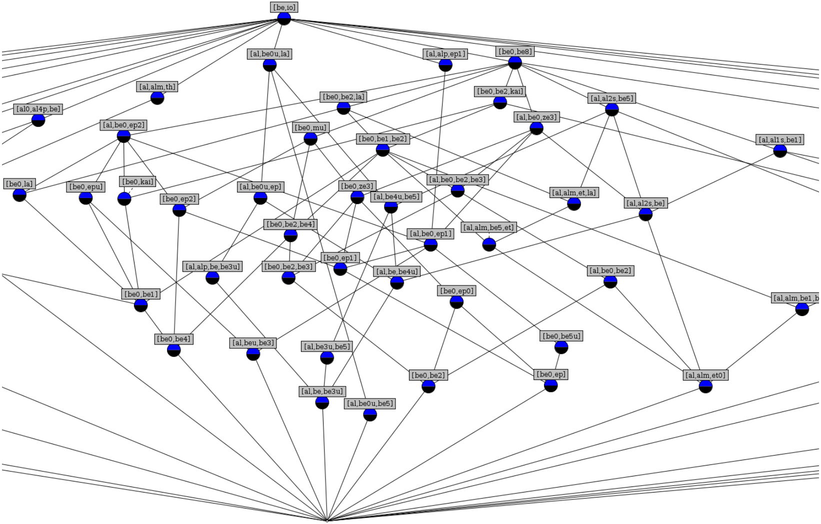

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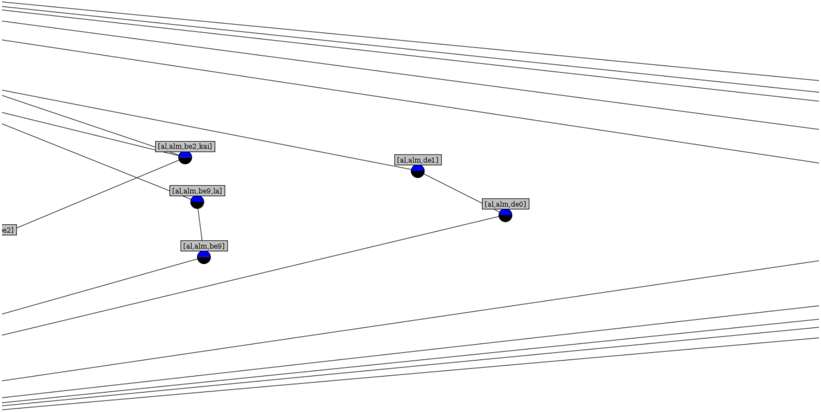
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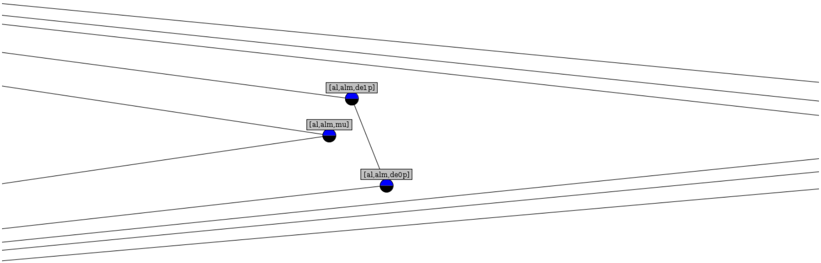

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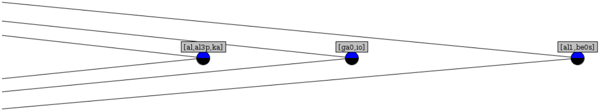
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