Problems on the frontier of commutator theory

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Novi Sad 2015

Outline

Modular
Commutator Theory

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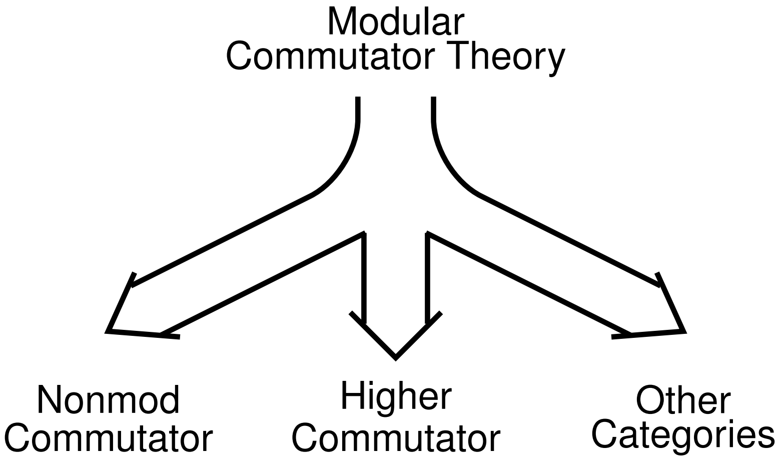
Modular
Commutator Theory
Nonmod
Commutator
Higher
Commutator
Other
Categories

New algebras in varieties generated by a class, 1

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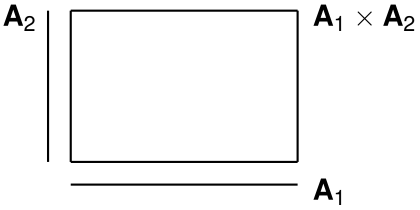

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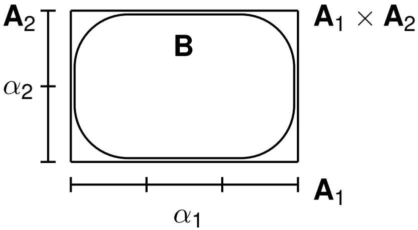

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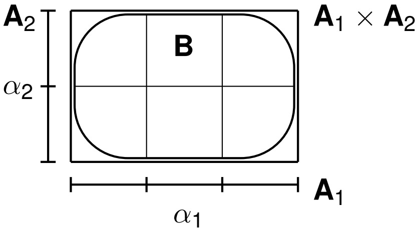

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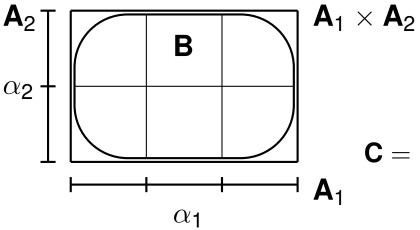

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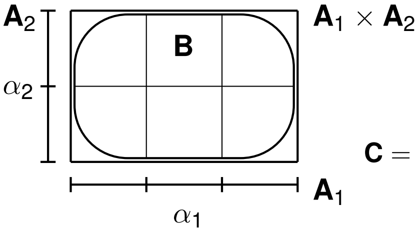

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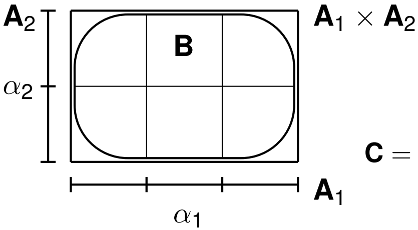

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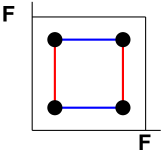
All other congruences are **skew**.

Examples

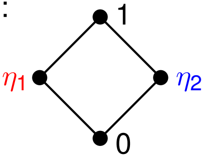
Field

$\mathbf{F} = (\{0, 1\}; +, \cdot, 0, 1)$

$\mathbf{F} \times \mathbf{F}$:



$\mathrm{Con}(\mathbf{F} \times \mathbf{F})$:

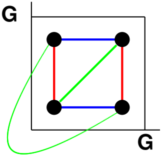


No skew congruence

Group

$\mathbf{G} = (\{0, 1\}; +, 0)$

$\mathbf{G} \times \mathbf{G}$:



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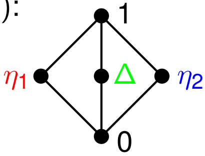

Δ is a (diagonal) skew congruence

Diagonal skew congruences, 1

$$\begin{array}{ccc} \mathbf{G} & \overset{\delta}{\hookrightarrow} & \mathbf{G} \times \mathbf{G} \\ \twoheaddownarrow & & \twoheaddownarrow \\ \mathbf{G}/\alpha & \hookrightarrow & (\mathbf{G} \times \mathbf{G})/\Delta \end{array}$$

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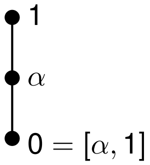
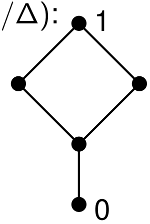
If $\mathbf{G} = \mathrm{Sl}_2(5)$, then

$\mathrm{Con}(\mathbf{G})$:



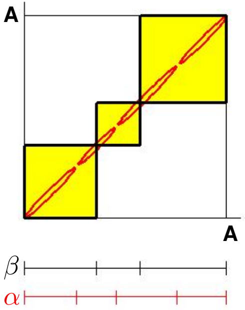
$\mathrm{Con}((\mathbf{G}\times\mathbf{G})/\Delta)$:



Diagonal Skew Congruences, 2

$\alpha, \beta \in \mathrm{Con}(\mathbf{A})$; $\mathbf{B} = \mathbf{A} \times_\beta \mathbf{A} := \beta \; (\leq \mathbf{A} \times \mathbf{A})$

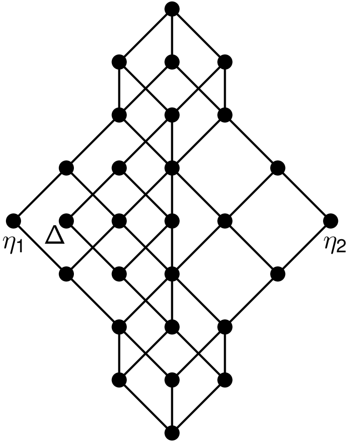
$\Delta_{\alpha,\beta}$=Δ= arising from pushout



$$\begin{array}{ccc} \mathbf{A} & \rightarrow & \mathbf{B} \\ \downarrow & & \\ \mathbf{A}/\alpha & & \end{array}$$

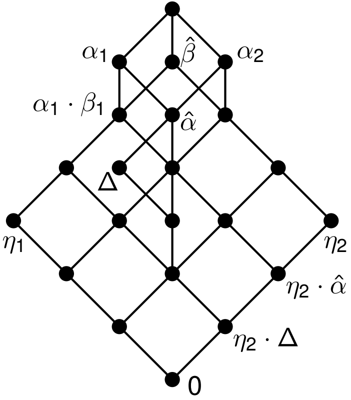
Where (in $\mathrm{Con}(\mathbf{A} \times_\beta \mathbf{A})$) is $\Delta_{\alpha,\beta}$?

$$\langle \eta_1,\ \eta_2,\ \Delta \mid \qquad\qquad \rangle$$



Where (in $\mathrm{Con}(\mathbf{A} \times_\beta \mathbf{A})$) is $\Delta_{\alpha,\beta}$?

$$\langle \eta_1,\ \eta_2,\ \Delta \mid \eta_1 \cdot \eta_2 = 0 \leq \Delta \rangle$$



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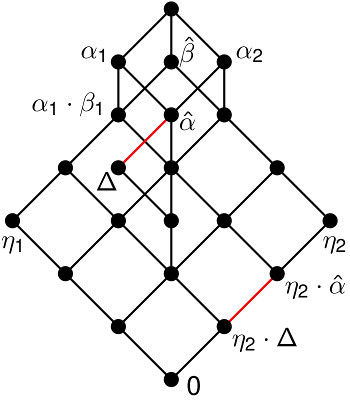

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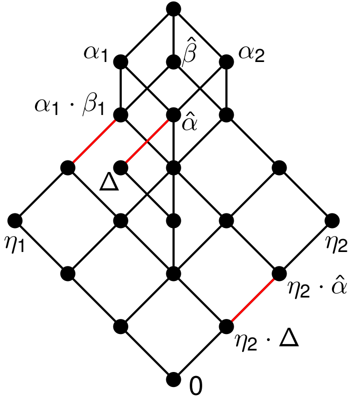

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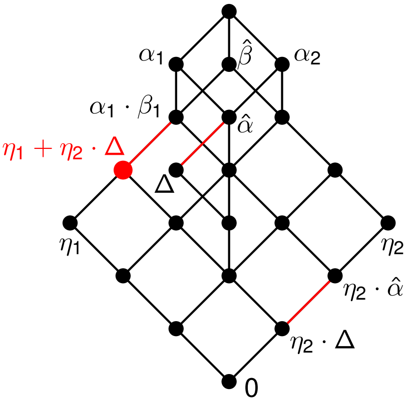
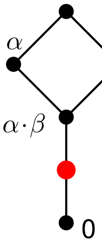
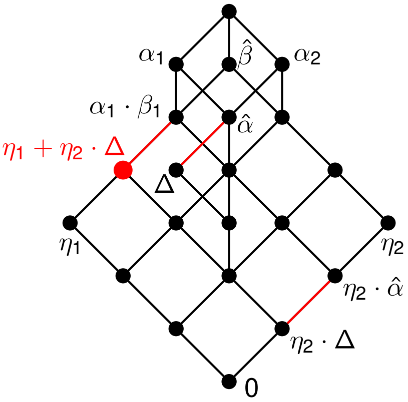


Where (in $\mathrm{Con}(\mathbf{A} \times_\beta \mathbf{A})$) is $\Delta_{\alpha,\beta}$?

$$\mathbf{A} \xleftarrow{\pi_1} \mathbf{A} \times_\beta \mathbf{A}$$

$$\mathrm{Con}(\mathbf{A}) \xleftarrow{\cong} I(\eta_1, 1)$$





Δ skew

$$\iff \Delta < (\Delta + \eta_1)(\Delta + \eta_2) = \hat{\alpha}$$

$$\iff \bullet < \alpha_1 \cdot \beta_1$$

$$\iff [\alpha, \beta] < \alpha \cdot \beta$$

$$I(\Delta, \hat{\alpha}) \searrow \nearrow I(\bullet, \alpha_1 \cdot \beta_1)$$

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Quasiaffine := subalgebra of a reduct of an affine algebra

Recall

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Commutator Theory

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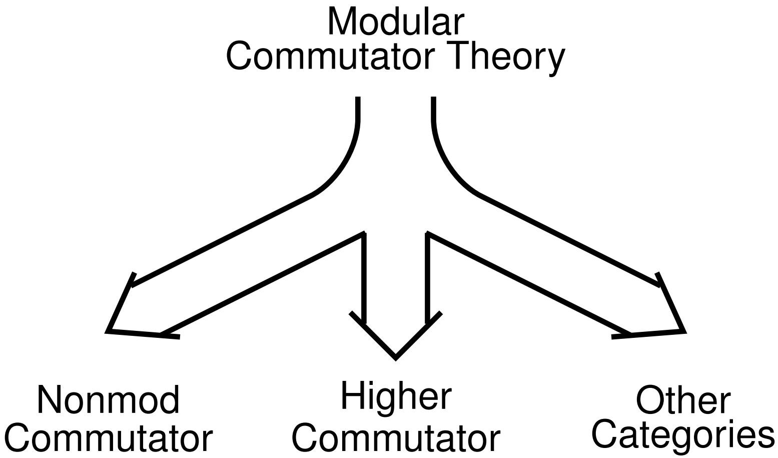
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3. **Which algebras are homomorphic images of finite abelian algebras?**

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7. **Does** $\exists$**weak difference term + symmetric commutator imply** $\exists$**difference term?**

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