On varietal joins of MV-algebras and some varieties

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Example: Let $(G, \leq, +, 0)$ be a partially ordered Abelian group and $u \in G^+$. Then $([0, u], +, 0, u)$ is an example of an effect algebra.

Definition: An effect algebra is a partial structure $(A, +, 0, 1)$ satisfying:

- $x + y = y + x$ if one side is defined;
- $(x + y) + z = x + (y + z)$ if one side is defined;
- for every x there is a unique x' (the orthosupplement of x) such that $x' + x = 1$;
- if $x + 1$ is defined, then $x = 0$.

Every EA has a natural partial order and subtraction:

- $x \leq y$ iff $y = x + z$, in which case $y - x := z$.

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Lattice effect algebras are EAs which are lattices w.r.t. $\leq$.

- In an orthomodular lattice $(A, \vee, \wedge, ', 0, 1)$, if we define

$$x + y := x \vee y \quad \text{iff} \quad x \leq y',$$

then $(A, +, 0, 1)$ is a LEA.

- OMLs are equivalent to LEAs satisfying $x \wedge x' = 0$, or equivalently, $x + x$ is defined iff $x = 0$.

- An MV-algebra is an algebra $(A, \oplus, ', 0, 1)$ such that $(A, \oplus, 0)$ is a commutative monoid satisfying $1 \oplus x = 1 = 0'$, $x'' = x$ and $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$.
  The natural lattice order is defined by $x \leq y$ iff $x' \oplus y = 1$.
- Every MV-algebra is isomorphic to one of the form $\Gamma(G, u) = ([0, u], \oplus, ', 0, u)$ where $(G, \leq, +, 0)$ is an Abelian ℓ-group, $u \in G^+$, $x \oplus y = (x + y) \wedge u$ and $x' = u - x$.
- In an MV-algebra $(A, \oplus, ', 0, 1)$, if we define

$$x + y := x \oplus y \quad \text{iff} \quad x \leq y',$$

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- Total operations on LEAs:

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$$x \oslash y := (x \vee y) - y = (x' \oplus y)' \quad \ldots \quad x \vee y = (x \oslash y) \oplus y$$
$$x \ominus y := x - (x \wedge y) = (y \oplus x')' \quad \ldots \quad x \wedge y = x \ominus (x \ominus y)$$

- LEAs are equivalent to algebras $(A, \oplus, ', 0, 1)$ of type $(2, 1, 0, 0)$ satisfying:

$$x \oplus 0 = x = x'',$$
$$(x \oslash y) \oplus y = (y \oslash x) \oplus x,$$
$$((x \oplus y) \oslash y) \oplus z \leq x \oplus z,$$
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For any algebra $(A, \oplus, ', 0, 1)$ in $\mathcal{E}$, let $(A, +, 0, 1)$ be defined as follows:

$$x + y \text{ exists iff } x \leq y', \text{ in which case } x + y := x \oplus y.$$

Then $(A, +, 0, 1)$ is a LEA with the same lattice order as $(A, \oplus, ', 0, 1)$, and $x \oplus y = (x \wedge y') + y$.

- The variety $\mathcal{E}$ is congruence distributive and regular.
- For $A \in \mathcal{E}$, $I \subseteq A$ is an ideal if $I = [0]_\theta$ for some $\theta \in \mathrm{Con}(A)$.
- The ideal lattice $\mathrm{Id}(A)$ is isomorphic to $\mathrm{Con}(A)$ under $\theta \mapsto [0]_\theta$ and $I \mapsto \theta_I$ where

$$(x, y) \in \theta_I \text{ iff } x \ominus y, y \ominus x \in I \text{ iff } x \oslash y, y \oslash x \in I.$$

- $\emptyset \neq I \subseteq A$ is an ideal iff
(i) $x \oplus y \in I$ for all $x, y \in I$;
(ii) $x \oslash y = (x \vee y) - y \in I$ for all $x \in I$, $y \in A$.

Compatibility, blocks, and sharp elements:

- In EAs, two elements a, b are compatible (in symbols: $a \leftrightarrow b$) if there exist a_1, b_1, c such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + c + b_1$ is defined.
- In LEAs, $a \leftrightarrow b$ iff $(a \vee b) - b = a - (a \wedge b)$ iff $(a \vee b) - b \leq a$ iff $a - (a \wedge b) \leq b'$.
- Hence, $a \leftrightarrow b$ iff $a \oslash b = a \ominus b$ iff $a \oslash b \leq a$ iff $a \ominus b \leq b'$ iff $a \oplus b = b \oplus a$ iff $a \leq a \oplus b$.
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- The compatibility center of A is

$$K(A) = \{a \in A \mid a \leftrightarrow x \text{ for all } x \in A\}.$$

- If $a \in K(A)$, then

$$\mathrm{Ig}(a) = \{x \in A \mid x \leq na \text{ for some } n \in \mathbb{N}\},$$

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An element $a \in A$ is sharp if $a \wedge a' = 0$; equivalently, $a \oplus a = a$. The sharp elements form a subalgebra $S(A)$ which is an OML in its own right.

$$\begin{array}{ccc} \text{MVAs} & \xrightarrow{x \wedge x' = 0} & \text{BAs} \\ \uparrow{\scriptstyle x \leftrightarrow y} & & \uparrow{\scriptstyle x \leftrightarrow y} \\ \text{LEAs} & \xrightarrow{x \wedge x' = 0} & \text{OMLs} \end{array}$$

Relative to $\mathcal{E}$,

- $\mathcal{MV}$ is axiomatized by $x \oplus y = y \oplus x$ or $x \leq x \oplus y$ or $x \oslash y \leq x$, and
- $\mathcal{OM}$, the subvariety of $\mathcal{E}$ term equivalent to the variety of OMLs, is axiomatized by $x \wedge x' = 0$ or $x \oplus x = x$.

What is the join $\mathcal{MV} \vee \mathcal{OM}$ in the lattice of subvarieties of $\mathcal{E}$?

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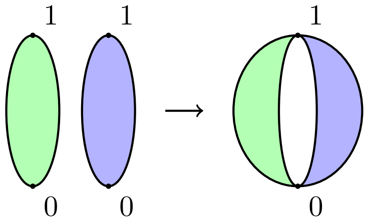
Horizontal sums of MV-algebras – simple members of $\mathcal{E}$:

Let $\{A_i \mid i \in I\}$ be a family of MV-algebras such that $A_i \cap A_j = \{0, 1\}$ for all $i \neq j$. The horizontal sum is the algebra $\bigoplus_{i \in I} A_i$ with domain $\bigcup_{i \in I} A_i$ on which the addition $\oplus$ is given by $x \oplus y = x \oplus_i y$ if there is $i \in I$ such that $x, y \in A_i$, and $x \oplus y = y$ otherwise.



Axiomatization of $\mathrm{V}(\mathcal{H}or)$:

Theorem

If $A \in \mathcal{E}$ is subdirectly irreducible, then the following are equivalent:

- A is the horizontal sum of its blocks;
- A satisfies the identity

$$(\gamma(x,y) \wedge z) \vee (\gamma(x,y)' \wedge z) = z, \tag{H}$$

  where $\gamma(x,y) = (x \ominus (x \oplus y)) \vee (y \ominus (y \oplus x))$.

Thus, relative to $\mathcal{E}$, $\mathrm{V}(\mathcal{H}or)$ is axiomatized by the identity (H).

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Relative to $\mathcal{E}$, $\mathcal{MV}$ and $\mathcal{OM}$ are axiomatized respectively by

$$x \oslash (x \oplus y) = 0 \quad \text{and} \quad x \wedge x' = 0.$$

Conjecture: $\mathcal{MV} \vee \mathcal{OM}$ is axiomatized by

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More generally, let $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{E}$ be axiomatized by $\sigma_1 = 0$ and $\sigma_2 = 0$, respectively. Is $\mathcal{V}_1 \vee \mathcal{V}_2$ axiomatized by $\sigma_1 \wedge \sigma_2 = 0$?

Problem: $a \wedge b = 0$ doesn't imply $\mathrm{Ig}(a) \cap \mathrm{Ig}(b) = \{0\}$, even when $a \leftrightarrow b \ldots$

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Lemma

Let $A \in \mathcal{E}$. Let $a \in A$ and suppose that

$$(x \oslash (x \oplus y)) \wedge a = 0$$

for all $x, y \in A$. Then $a \in K(A)$ and the polar

$$a^{\perp} = \{x \in A \mid x \wedge a = 0\}$$

is an ideal of A such that $\mathrm{Ig}(a) \cap a^{\perp} = \{0\}$.

Theorem

Let $\mathcal{U}$ be a subvariety of $\mathcal{E}$, incomparable with $\mathcal{MV}$, which is axiomatized, relative to $\mathcal{E}$, by $\tau(x_1, \ldots, x_n) = 0$. Then the join $\mathcal{MV} \vee \mathcal{U}$ is axiomatized by

$$(x \oslash (x \oplus y)) \wedge \tau(z_1, \ldots, z_n) = 0. \qquad \text{(J)}$$

Proof: Let $\mathcal{W}$ be the subvariety of $\mathcal{E}$ defined by the identity (J). Then $\mathcal{MV} \vee \mathcal{U} \subseteq \mathcal{W}$.

Conversely, let $A \in \mathrm{Si}(\mathcal{W}) \setminus \mathcal{U}$ and take $a_1, \ldots, a_n \in A$ so that $a := \tau(a_1, \ldots, a_n) \neq 0$. Then $(x \oslash (x \oplus y)) \wedge a = 0$ for all $x, y \in A$, hence by the lemma, $a \in K(A)$, $a^\perp$ is an ideal and $\mathrm{Ig}(a) \cap a^\perp = \{0\}$. Since $\mathrm{Ig}(a) \neq \{0\}$, $a^\perp = \{0\}$ and so $x \oslash (x \oplus y) = 0$ for all $x, y \in A$. Thus $A \in \mathcal{MV}$ and we have $\mathcal{W} = \mathcal{MV} \vee \mathcal{U}$.

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Corollary

Relative to $\mathcal{E}$, the join $\mathcal{MV} \vee \mathcal{OM}$ is axiomatized by

$$(x \oslash (x \oplus y)) \wedge z \wedge z' = 0.$$

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Thank you!