Algebra and the complexity of quantified constraints

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The **Constraint Satisfaction Problem** CSP(\(B\)) takes as input a *primitive positive* (pp) sentence \(\Phi\), i.e. of the form

\[
\exists v_1 \ldots v_j \phi(v_1, \ldots, v_j),
\]

where \(\phi\) is a conjunction of atoms, and asks whether \(B \models \Phi\).

This is equivalent to the **Homomorphism Problem** – has \(A\) a homomorphism to \(B\)?

The structure \(B\) is known as the **template**.
Finite CSPs occur a lot in nature.

- CSP($\mathcal{K}_m$) is graph $m$-colourability.
- CSP($\{0, 1\}; R_{NAE}$), where $B_{NAE}$ is $\{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$ is not-all-equal 3-satisfiability.
- CSP($\{0, 1\}; R_{TTT}, R_{TTF}, R_{TFF}, R_{FFF}$) is 3-satisfiability.
- CSP($\{0, 1\}; \{0\}, \{1\}, \{(0, 0), (1, 1)\}$) is graph s-t unreachability.

Also vertex cover, clique and hamilton path – but these require non-fixed template.

Infinite CSPs also occur a lot in nature (another story...)
Feder-Vardi **dichotomy** conjecture. Each CSP(\(\mathcal{B}\)) is either in P or is NP-complete.

- Compare with Ladner **non-dichotomy** for NP.

Still open, but known for:

- Structures size 2 (Schaefer 1978).
- Structures size 3 (Bulatov 2002).
- Structures with unary relations (Bulatov 2003).
- Smooth digraphs (Barto, Kozik and Niven 2010).
- Structures size 4 (Marković 2011?).
Manuel Bodirsky calls the CSP *Königsproblem* because it is a beautiful marriage of

- **logic** (primitive positive model theory)
- **combinatorics** (structure homomorphism)
- **algebra** (polymorphism clones and varieties)

to an important class of problems in computer science.
The **Quantified CSP** QCSP(\(B\)) takes as input a **positive Horn (pH)** sentence \(\Phi\), i.e. of the form

\[
\forall \overline{v}_1 \exists \overline{v}_2 \ldots, Q \overline{v}_j \phi(\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_j),
\]

where \(\phi\) is a conjunction of atoms, and asks whether \(B \models \Phi\).

QCSP(\(B\)) is always in Pspace.
- QCSPs used in AI to model non-monotonic reasoning.
Previous classifications

QCSP classifications are harder than CSP classifications.

- **Boolean structures.** *Dichotomy* P, Pspace-complete. (Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.)
- **Graphs of permutations.** *Trichotomy* P, NP-complete, Pspace-complete. (Börner et al. 2002.)
- **Various digraphs** *Dichotomies and trichotomies* NL, NP-complete, Pspace-complete. (Madelaine, M. 2006, 2011, 2013; Dapić, Marković, M. 2014 etc.)
- **Structures with 2-semilattice polymorphism.** *Dichotomy* P, coNP-hard. (Chen 2004.)
The recent advances in CSP complexity classification are due to the algebraic approach.

- A $k$-ary polymorphism of $\mathcal{B}$ is a homomorphism from $\mathcal{B}^k$ to $\mathcal{B}$.

The key to this approach is the Galois correspondence

$$\text{Inv}(\text{Pol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{\text{pp}}$$

whose consequence is

$$\text{Pol}(\mathcal{B}) \subseteq \text{Pol}(\mathcal{B}') \Rightarrow \text{CSP}(\mathcal{B}') \leq_{\text{P}} \text{CSP}(\mathcal{B})$$
The algebraic approach exists also for the QCSP.

\[ \text{Inv}(s\text{Pol}(\mathcal{B})) = \langle \mathcal{B} \rangle_{PH} \]

whose consequence is

\[ s\text{Pol}(\mathcal{B}) \subseteq s\text{Pol}(\mathcal{B}') \Rightarrow \text{QCSP}(\mathcal{B}') \leq \text{P QCSP}(\mathcal{B}). \]

It appears to be weaker (surjective operations are not closed under composition) and we have fewer combinatorial constructs.

- Important?
- Königsproblem?

Assume henceforth that finite \( \mathcal{B} \) contains constants naming each element. Now all polymorphisms are idempotent and \( s\text{Pol}(\mathcal{B}) = \text{Pol}(\mathcal{B}) \).
Following Chen, for $z \in B$, call $B$ logically $k$-collapsible from source $\{z\}$ if truth of a pH sentence can be decided by sub-sentences in which all but $k$ universal variables are forced to $z$.

E.g. $k := 2$ and for the pH sentence

$$\forall x_1 \forall x_2 \exists y_1 \forall x_3 \forall x_4 \exists y_2 \ E(x_1, y_1) \land E(x_2, y_1) \land E(x_3, y_2) \land E(x_4, y_2),$$

we obtain the 2-collapsings

$$\forall x_1 \forall x_2 \exists y_1 \exists y_2 \ E(x_1, y_1) \land E(x_2, y_1) \land E(z, y_2) \land E(z, y_2)$$
$$\forall x_1 \exists y_1 \forall x_3 \exists y_2 \ E(x_1, y_1) \land E(z, y_1) \land E(x_3, y_2) \land E(z, y_2)$$
$$\forall x_1 \exists y_1 \forall x_4 \exists y_2 \ E(x_1, y_1) \land E(z, y_1) \land E(z, y_2) \land E(x_4, y_2)$$
$$\forall x_2 \exists y_1 \forall x_3 \exists y_2 \ E(z, y_1) \land E(x_2, y_1) \land E(x_3, y_2) \land E(z, y_2)$$
$$\forall x_2 \exists y_1 \forall x_4 \exists y_2 \ E(z, y_1) \land E(x_2, y_1) \land E(z, y_2) \land E(x_4, y_2)$$
$$\exists y_1 \forall x_3 \exists x_5 \exists y_2 \ E(z, y_1) \land E(z, y_1) \land E(x_3, y_2) \land E(x_4, y_2)$$
If $\mathcal{B}$ is logically $k$-collapsible, then QCSP($\mathcal{B}$) “collapses” to an ensemble of instances of CSP($\mathcal{B}$) and QCSP($\mathcal{B}$) is in NP.

Call an idempotent clone $\mathcal{B}$

- *algebraically $k$-collapsible* from source $\{z\}$

if it contains $f$ so that for each $m$, the image under $f$ of set tuples that are co-ordinate permutations of

$$k \text{ times } m-k \text{ times}$$

$$\begin{cases} B, \ldots, B, \\ \{z\}, \ldots, \{z\} \end{cases}$$

is

$$m \text{ times}$$

$$\begin{cases} B, \ldots, B \end{cases}.$$
E.g. \( m := 4, k := 2 \).

\[
\begin{align*}
\{z\} & \quad \{z\} & \quad \{z\} & \quad B & \quad B & \quad B & \quad B \\
\{z\} & \quad B & \quad B & \quad \{z\} & \quad \{z\} & \quad B & \quad f & \quad B \\
B & \quad \{z\} & \quad B & \quad \{z\} & \quad B & \quad \{z\} & \quad \rightarrow & \quad B \\
B & \quad B & \quad \{z\} & \quad B & \quad \{z\} & \quad \{z\} & \quad B
\end{align*}
\]

**Theorem (Chen 2006)**

*If \( \text{Pol}(B) \) is algebraically \( k \)-collapsible from source \( Z \), then \( B \) is logically \( k \)-collapsible from source \( Z \).*

**Theorem (Carvalho, Madelaine, M. 2015)**

*If \( B \) is logically \( k \)-collapsible from source \( Z \), then \( \text{Pol}(B) \) is algebraically \( k \)-collapsible from source \( Z \).*
In many QCSP classifications, all NP memberships can be explained uniformly by collapsibility. For example, this is true of all the classifications we already saw. But,

**Theorem (Chen 2008)**

*There is $\mathcal{B}$ on 3-elements so that $\text{Pol}(\mathcal{B})$ is “switchable” but not collapsible, and $\text{QCSP}(\mathcal{B})$ is in NP.*
Collapsibility looks like a form of the polynomial generated powers property (PGP): E.g. \( m := 4, k := 2 \).

\[
\begin{array}{cccccc}
\{z\} & \{z\} & \{z\} & B & B & B \\
\{z\} & B & B & \{z\} & \{z\} & B & f & B \\
B & \{z\} & B & \{z\} & B & \{z\} & \rightarrow & B \\
B & B & \{z\} & B & \{z\} & \{z\} & B \\
\end{array}
\]

Imagine for \(|B| = 2\) that each column becomes

\[
\begin{array}{cccccc}
\{z\} & z & z & z & z \\
\{z\} & \Rightarrow & z & z & z & z \\
B & \Rightarrow & b_1 & b_1 & b_2 & b_2 \\
B & b_1 & b_2 & b_1 & b_2 \\
\end{array}
\]

and for each \(x_1, x_2, x_3, x_4 \in B\) there exists an \(f\) (i.e. this function is no longer uniform).
This is saying that $B^m$ is generated, in $\text{Pol}(B)$, from the set of tuples of the form of co-orindate permutations of $(b_1, \ldots, b_k, z, \ldots, z)$, a set that we call $C^m_{\{z\}}$.

**Theorem (Carvalho, Madelaine, M. 2015)**

$\text{Pol}(B)$ is algebraically $k$-collapsible from source $Z$ iff, for all $m$, $B^m$ is generated in $\text{Pol}(B)$ by $C^m_Z$.

**Message:** Collapsibility well understood in idempotent singleton source case; and quite well understood in general idempotent case.

**Conjecture (Chen)**

$\text{QCSP}(B)$ in $NP$ iff $\text{Pol}(B)$ has the PGP; and Pspace-complete otherwise.