Rank properties of transformation semigroups with restricted range

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     - Rank($T(\mathbb{N}, Y) : O(\mathbb{N}, Y)$)
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     - Rank($T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)$)
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Background


Introduction and Preliminaries

Let $X$ be a non-empty set and $Y \subseteq X$. Then a mapping $\alpha$ from set $X$ to set $X$ is called **full transformation**.

Then $T(X)$ is the **full transformation semigroup** under composition.

The semigroup $T(X, Y)$ was introduced by J.S.V. Symons in 1975, called **semigroup with restricted range**. If $X = Y$ then $T(X, Y)$ is the full transformation semigroup $T(X)$. 
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Let \( X \) be a non-empty set and \( Y \subseteq X \). Then a mapping \( \alpha \) from set \( X \) to set \( X \) is called full transformation. Then \( T(X) \) is the full transformation semigroup under composition.

The semigroup \( T(X, Y) \) was introduced by J.S.V. Symons in 1975, called semigroup with restricted range. If \( X = Y \) then \( T(X, Y) \) is the full transformation semigroup \( T(X) \).
Let $X$ be an infinite chain and $Y$ be a subchain of $X$. Let

$$O(X) = \{ \alpha \in T(X) : \forall x, y \in X, x \leq y \Rightarrow x\alpha \leq y\alpha \}$$

be the set of all order-preserving transformations. We define

$$O(X, Y) := O(X) \cap T(X, Y)$$

For a semigroup $S$, the idea of rank is concerned with finding minimum size generating sets for $S$, denoted $\text{rank}(S)$. 
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For a semigroup $S$, the idea of rank is concerned with finding minimum size generating sets for $S$, denoted $\text{rank}(S)$. 
For a semigroup $S$, if $A \subseteq S$ then we call the minimum cardinality of a set $B$ such that $\langle A \cup B \rangle = S$ the relative rank of $S$ modulo $A$, denoted $\text{rank}(S : A)$.

Let $\alpha \in T(X, Y)$.

- The set $X_\alpha$ is called image of $\alpha$, denoted $\text{im}\alpha$.

- The cardinality of the set $\text{im}\alpha$ is called rank of $\alpha$, denoted $\text{rank}\alpha := |\text{im}\alpha|$.
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- The cardinality of the set $\text{im} \alpha$ is called rank of $\alpha$, denoted $\text{rank} \alpha := |\text{im} \alpha|$. 
The kernel of $\alpha$, denoted $\ker \alpha := \{(x, y) \in X \times X : x\alpha = y\alpha\}$, is equivalence relation on $X$, which corresponds uniquely to a decomposition of $X$ into blocks, called $\ker \alpha$-classes.

The infinite contraction index define as the number of classes of $\ker \alpha$ of size $|X|$.
The kernel of \( \alpha \), denoted
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is equivalence relation on \( X \), which corresponds uniquely to a decomposition of \( X \) into blocks, called \( \ker \alpha \)-classes.

The infinite contraction index define as the number of classes of \( \ker \alpha \) of size \( |X| \).
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     • \text{Rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y))
Main Results

**General Results**

\[
\text{rank}(T(X, Y) : O(X, Y))
\]

Let \( X \) be an infinite chain and \( Y \) be a non-empty subchain of \( X \) such that \( |Y| = |X| \) or \( Y \) is finite.

**Proposition 1**

Let \( X \) be an infinite chain and \( Y \) be a subchain of \( X \) such that there exists \( X' \subseteq Y \) such that \( |X'| = |Y| = |X| \) and where any order-preserving map from \( X' \) to \( X' \) can be extended to an order-preserving map from \( X \) to \( Y \). Then

\[
\text{rank}(T(X') : O(X')) \leq 2 \implies \text{rank}(T(X, Y) : O(X, Y)) \leq 2.
\]

**Corollary 1**

Let \( X \) be an infinite chain such that there are a well ordered subchain \( X' \subseteq Y \) with \( |X'| = |Y| = |X| \) and \( a \in Y \) with \( a > x' \) for all \( x' \in X' \). Then

\[
\text{rank}(T(X, Y) : O(X, Y)) \leq 2.
\]
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General Results

**rank**($T(X, Y) : O(X, Y)$)

Let $X$ be an infinite chain and $Y$ be a non-empty subchain of $X$ such that $|Y| = |X|$ or $Y$ is finite.

**Proposition 1**

Let $X$ be an infinite chain and $Y$ be a subchain of $X$ such that there exists $X' \subseteq Y$ such that $|X'| = |Y| = |X|$ and where any order-preserving map from $X'$ to $X'$ can be extended to an order-preserving map from $X$ to $Y$. Then $\text{rank}(T(X') : O(X')) \leq 2$ implies $\text{rank}(T(X, Y) : O(X, Y)) \leq 2$.

**Corollary 1**

Let $X$ be an infinite chain such that there are a well ordered subchain $X' \subseteq Y$ with $|X'| = |Y| = |X|$ and $a \in Y$ with $a > x'$ for all $x' \in X'$. Then $\text{rank}(T(X, Y) : O(X, Y)) \leq 2$. 
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Main Results

General Results

\[ \text{rank}(T(X, Y) : O(X, Y)) \]

Next, we define \((x] := \{ y \in Y : y \leq x \}\) on the chain \(Y\) for \(x \in Y\).

**Proposition 2**

Let \(X\) be an infinite chain and \(Y\) be a subchain of \(X\) with \(|Y| = |X|\). If there is a well ordered subset \(X' \subseteq Y\) such that \(|X'| = |X|\), \(\{x \in Y : x > X'\} = \emptyset\), \(|(x]| < |X|\) for all \(x \in Y\), and there exists \(z \in X\) with \(z > y\) for all \(y \in Y\), then \(\text{rank}(T(X, Y) : O(X, Y))\) is infinite.

**Proposition 3**

Let \(X\) be an infinite set and \(Y \subseteq X\) be a finite set with at least two elements. Then \(\text{rank}(T(X, Y) : O(X, Y))\) is infinite.
**Main Results**

**General Results**

\[
\text{rank}(T(X, Y) : O(X, Y))
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Next, we define \((x] := \{y \in Y : y \leq x\}\) on the chain \(Y\) for \(x \in Y\).

**Proposition 2**

Let \(X\) be an infinite chain and \(Y\) be a subchain of \(X\) with \(|Y| = |X|\). If there is a well ordered subset \(X' \subseteq Y\) such that \(|X'| = |X|\), \(\{x \in Y : x > X'\} = \emptyset\), \(|(x]| < |X|\) for all \(x \in Y\), and there exists \(z \in X\) with \(z > y\) for all \(y \in Y\), then \(\text{rank}(T(X, Y) : O(X, Y))\) is infinite.

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Main Results

General Results

\( \text{rank}(T(X, Y) : O(X, Y)) \)

Lemma 1

Let \( X \in \{\mathbb{N}, \mathbb{Z}\} \) and \( Y \) be a countable subchain of \( X \) with minimum element. If \( \delta \in T(X, Y) \) such that \( k(\delta|_Y) = |X| \), then there is \( \gamma \in \langle O(X, Y), \delta \rangle \) such that \( |B \cap Y| = |X| \) for all \( B \in \ker \gamma \).
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$$\text{Rank}(T(\mathbb{N}, Y) : O(\mathbb{N}, Y))$$

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**Proposition 4**

Let $Y$ be a non-empty countable subset of $\mathbb{N}$. Then
\[
\langle O(\mathbb{N}, Y), \delta \rangle = T(\mathbb{N}, Y) \text{ such that } \delta \in T(\mathbb{N}, Y) \text{ with } k(\delta|_Y) \text{ is infinite, i.e. } \text{rank}(T(\mathbb{N}, Y) : O(\mathbb{N}, Y)) = 1.
\]

**Theorem 1**

Let $Y$ be a non-empty countable subset of $\mathbb{N}$ and let $\delta \in T(\mathbb{N}, Y)$. Then
\[
\langle O(\mathbb{N}, Y), \delta \rangle = T(\mathbb{N}, Y) \text{ if and only if } k(\delta|_Y) = |\mathbb{N}|.
\]
Main Results

\[ \text{Rank}(T(\mathbb{N}, Y) : O(\mathbb{N}, Y)) \]

**Proposition 4**

Let \( Y \) be a non-empty countable subset of \( \mathbb{N} \). Then
\[ \langle O(\mathbb{N}, Y), \delta \rangle = T(\mathbb{N}, Y) \] such that \( \delta \in T(\mathbb{N}, Y) \) with \( k(\delta|_Y) \) is infinite, i.e. \( \text{rank}(T(\mathbb{N}, Y) : O(\mathbb{N}, Y)) = 1 \).

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Let \( Y \) be a non-empty countable subset of \( \mathbb{N} \) and let \( \delta \in T(\mathbb{N}, Y) \). Then \( \langle O(\mathbb{N}, Y), \delta \rangle = T(\mathbb{N}, Y) \) if and only if \( k(\delta|_Y) = |\mathbb{N}| \).
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\[ \text{Rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) \]

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\[ \text{Rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) \]

Proposition 5

Let \( Y \) be a non-empty subset of \( \mathbb{Z} \) with \( |Y| = |\mathbb{Z}| \). If \( Y \) has minimum element or maximum element then
\[ \text{rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) > 1, \text{ i.e. } \langle O(\mathbb{Z}, Y), \delta \rangle \neq T(\mathbb{Z}, Y) \text{ where } \delta \in T(\mathbb{Z}, Y) \setminus O(\mathbb{Z}, Y). \]

Proposition 6

Let \( Y \) be a non-empty subset of \( \mathbb{Z} \) with \( |Y| = |\mathbb{Z}| \). If \( Y \) has minimum element or maximum element then
\[ \text{rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) \leq 2. \]

Corollary 2

\[ \text{rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) = 2. \]
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\[ \text{Rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) \]

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**Theorem 2**

Let $Y$ be a non-empty subset of $\mathbb{Z}$ with $Y$ has minimum element or maximum element such that $|Y| = |\mathbb{Z}|$ and let $\varphi, \delta \in T(\mathbb{Z}, Y)$. Then $\langle O(\mathbb{Z}, Y), \varphi, \delta \rangle = T(\mathbb{Z}, Y)$ if and only if $\varphi$ is injective and $k(\delta|_Y) = |\mathbb{Z}|$.

**Proposition 7**

Let $Y$ be a non-empty subset of $\mathbb{Z}$ with $|Y| = |\mathbb{Z}|$. If $Y$ has neither maximum element nor minimum element, then $\text{rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) = 1$. 

\[ \text{rank}(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) \]
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**Theorem 2**

Let $Y$ be a non-empty subset of $\mathbb{Z}$ with $Y$ has minimum element or maximum element such that $|Y| = |\mathbb{Z}|$ and let $\varphi, \delta \in T(\mathbb{Z}, Y)$. Then $\langle O(\mathbb{Z}, Y), \varphi, \delta \rangle = T(\mathbb{Z}, Y)$ if and only if $\varphi$ is injective and $k(\delta|_Y) = |\mathbb{Z}|$.

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\[ \text{Rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \]

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Main Results

\[ \text{Rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \]

**Proposition 8**

Let \( Y \subseteq \mathbb{Q} \) with \( |Y| = |\mathbb{Q}| \) such that there is \( a \in Y \) with \( \{y \in Y : y \leq a\} \) is infinite or there is no \( x \in \mathbb{Q} \) with \( x > y \) for all \( y \in Y \). Then \( \text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2 \).

**Proposition 9**

Let \( Y \subseteq \mathbb{Q} \) with \( |Y| = |\mathbb{Q}| \) such that there is \( a \in Y \) with \( \{y \in Y : y \leq a\} \) is infinite or there is no \( x \in \mathbb{Q} \) with \( x > y \) for all \( y \in Y \). Then the following statements are equivalent:

1. \( \text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2 \),
2. \( |Y| = |\mathbb{Q}| \).
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\[ \text{Rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \]

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1. \( \text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2 \),
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\[ \text{Rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \]

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Let \( X = \mathbb{Q} \). For \( a, b \in X \) with \( a < b \), we define

\[ [a, b] = \{ z \in X : a \leq z \leq b \}, \]
\[ (a, b) = \{ z \in X : a < z < b \}, \]
\[ (a, b] = \{ z \in X : a < z \leq b \}, \]
\[ [a, b) = \{ z \in X : a \leq z < b \}. \]

We call these sets intervals that delimited by \( a \) and \( b \).

**Proposition 10**

Let \( Y \subseteq \mathbb{Q} \) be any interval. Then \( \text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2 \).

**Theorem 3**

Let \( Y = [a, b] \subseteq \mathbb{Q} \) for some \( a, b \in \mathbb{Q} \) such that \( a < b \) and \( |Y| = |\mathbb{Q}| \). Then \( \text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) = 1 \).
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\[ \text{Rank}(T(Q, Y) : O(Q, Y)) \]

\begin{align*}
\text{rank}(T(Q, Y) : O(Q, Y)) & \\
\text{Let } X = \mathbb{Q}. \text{ For } a, b \in X \text{ with } a < b, \text{ we define} & \\
[a, b] &= \{ z \in X : a \leq z \leq b \}, (a, b) = \{ z \in X : a < z < b \}, \\
(a, b] &= \{ z \in X : a < z \leq b \}, [a, b) = \{ z \in X : a \leq z < b \}. & \\
\text{We call these sets intervals that delimited by } a \text{ and } b. & \\
\end{align*}

Proposition 10

Let } Y \subseteq \mathbb{Q} \text{ be any interval. Then } \text{rank}(T(Q, Y) : O(Q, Y)) \leq 2.

Theorem 3

Let } Y = [a, b] \subseteq \mathbb{Q} \text{ for some } a, b \in \mathbb{Q} \text{ such that } a < b \text{ and } |Y| = |\mathbb{Q}|. \text{ Then } \text{rank}(T(Q, Y) : O(Q, Y)) = 1.
Main Results

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\textbf{Proposition 10}

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Main Results

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\text{rank}(T(Q, Y) : O(Q, Y))
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THANK YOU FOR YOUR ATTENTION.