Rank properties of transformation semigroups with restricted range

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Outline

Background

- In 2013, Fernandes, V.H., Honyam, P., Quinteiro, T.M., Singha, B., *On semigroups of endomorphisms of a chain with restriced range*, Semigroup Forum **89**, 77-104.

- In 1992, Gomes, G.M.S., Howie, J.M., *On the ranks of certain semigroups of order-preserving transformations*, Semigroup Forum **45** , 272-282.

- In 1998, Howie, J.M., Ruskuc, N., Higgins, P.M., *On relative ranks of full transformation semigroups*, Communications in Algebra **26(3)** , 733-748.

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Introduction and Preliminaries

Let X be a non-empty set and $Y \subseteq X$. Then a mapping α from set X to set X is called **full transformation.**

Then $T(X)$ is the full transformation semigroup under composition.

The semigroup $T(X, Y)$ was introduced by J.S.V. Symons in 1975, called semigroup with restricted range. If $X = Y$ then $T(X, Y)$ is the full transformation semigroup $T(X)$.

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Let X be an infinite chain and Y be a subchain of X. Let

$$O(X) = \{\alpha \in T(X) : \forall x, y \in X, x \leq y \Rightarrow x\alpha \leq y\alpha\}$$

be the set of all order-preserving transformations. We define

$$O(X, Y) := O(X) \cap T(X, Y)$$

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For a semigroup S, if $A \subseteq S$ then we call the minimum cardinality of a set B such that $\langle A \cup B \rangle = S$ the relative rank of S modulo A, denoted $rank(S : A)$.

Let $\alpha \in T(X, Y)$.

- The set $X\alpha$ is called image of α, denoted $im\alpha$.
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- The kernel of α, denoted $\ker\alpha := \{(x,y) \in X \times X : x\alpha = y\alpha\}$, is equivalence relation on X, which corresponds uniquely to a decomposition of X into blocks, called $\ker\alpha$-classes.

- The infinite contraction index define as the number of classes of $\ker\alpha$ of size $|X|$.

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$rank(T(X,Y):O(X,Y))$

Let X be an infinite chain and Y be a non-empty subchain of X such that $|Y|=|X|$ or Y is finite.

Proposition 1

Let X be an infinite chain and Y be a subchain of X such that there exists $X' \subseteq Y$ such that $|X'|=|Y|=|X|$ and where any order-preserving map from X' to X' can be extended to an order-preserving map from X to Y. Then $rank(T(X'):O(X')) \leq 2$ implies $rank(T(X,Y):O(X,Y)) \leq 2$.

Corollary 1

Let X be an infinite chain such that there are a well ordered subchain $X' \subseteq Y$ with $|X'|=|Y|=|X|$ and $a \in Y$ with $a > x'$ for all $x' \in X'$. Then $rank(T(X,Y):O(X,Y)) \leq 2$.

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Let X be an infinite chain and Y be a non-empty subchain of X such that $|Y| = |X|$ or Y is finite.

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Next, we define $(x] := \{y \in Y : y \leq x\}$ on the chain Y for $x \in Y$.

Proposition 2

Let X be an infinite chain and Y be a subchain of X with $|Y| = |X|$. If there is a well ordered subset $X' \subseteq Y$ such that $|X'| = |X|$, $\{x \in Y : x > X'\} = \emptyset$, $|(x]| < |X|$ for all $x \in Y$, and there exists $z \in X$ with $z > y$ for all $y \in Y$, then $rank(T(X,Y):O(X,Y))$ is infinite.

Proposition 3

Let X be an infinite set and $Y \subseteq X$ be a finite set with at least two elements. Then $rank(T(X,Y):O(X,Y))$ is infinite.

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$rank(T(X,Y) : O(X,Y))$

Lemma 1

Let $X \in \{\mathbb{N}, \mathbb{Z}\}$ and Y be a countable subchain of X with minimum element. If $\delta \in T(X,Y)$ such that $k(\delta|_Y) = |X|$, then there is $\gamma \in \langle O(X,Y), \delta \rangle$ such that $|B \cap Y| = |X|$ for all $B \in \ker \gamma$.

Outline

rank$(T(\mathbb{N}, Y) : O(\mathbb{N}, Y))$

Proposition 4

Let Y be a non-empty countable subset of $\mathbb{N}$. Then $\langle O(\mathbb{N}, Y), \delta\rangle = T(\mathbb{N}, Y)$ such that $\delta \in T(\mathbb{N}, Y)$ with $k(\delta|_Y)$ is infinite, i.e. rank$(T(\mathbb{N}, Y) : O(\mathbb{N}, Y)) = 1$.

Theorem 1

Let Y be a non-empty countable subset of $\mathbb{N}$ and let $\delta \in T(\mathbb{N}, Y)$. Then $\langle O(\mathbb{N}, Y), \delta\rangle = T(\mathbb{N}, Y)$ if and only if $k(\delta|_Y) = |\mathbb{N}|$.

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Proposition 5

Let Y be a non-empty subset of $\mathbb{Z}$ with $|Y| = |\mathbb{Z}|$. If Y has minimum element or maximum element then rank$(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) > 1$, i.e. $\langle O(\mathbb{Z}, Y), \delta\rangle \neq T(\mathbb{Z}, Y)$ where $\delta \in T(\mathbb{Z}, Y) \setminus O(\mathbb{Z}, Y)$.

Proposition 6

Let Y be a non-empty subset of $\mathbb{Z}$ with $|Y| = |\mathbb{Z}|$. If Y has minimum element or maximum element then rank$(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) \leq 2$.

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Let Y be a non-empty subset of $\mathbb{Z}$ with Y has minimum element or maximum element such that $|Y| = |\mathbb{Z}|$ and let $\varphi, \delta \in T(\mathbb{Z}, Y)$. Then $\langle O(\mathbb{Z}, Y), \varphi, \delta \rangle = T(\mathbb{Z}, Y)$ if and only if φ is injective and $k(\delta|_Y) = |\mathbb{Z}|$.

Proposition 7

Let Y be a non-empty subset of $\mathbb{Z}$ with $|Y| = |\mathbb{Z}|$. If Y has neither maximum element nor minimum element, then $rank(T(\mathbb{Z}, Y) : O(\mathbb{Z}, Y)) = 1$.

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rank$(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y))$

Proposition 8

Let $Y \subseteq \mathbb{Q}$ with $|Y| = |\mathbb{Q}|$ such that there is $a \in Y$ with $\{y \in Y : y \leq a\}$ is infinite or there is no $x \in \mathbb{Q}$ with $x > y$ for all $y \in Y$. Then rank$(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2$.

Proposition 9

Let $Y \subseteq \mathbb{Q}$ with $|Y| = |\mathbb{Q}|$ such that there is $a \in Y$ with $\{y \in Y : y \leq a\}$ is infinite or there is no $x \in \mathbb{Q}$ with $x > y$ for all $y \in Y$. Then the following statements are equivalent:

1. *rank$(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2$,*
2. $|Y| = |\mathbb{Q}|$.

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Let $X = \mathbb{Q}$. For $a, b \in X$ with $a < b$, we define

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We call these sets intervals that delimited by a and b.

Proposition 10

Let $Y \subseteq \mathbb{Q}$ be any interval. Then $\text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) \leq 2$.

Theorem 3

Let $Y = [a, b] \subseteq \mathbb{Q}$ for some $a, b \in \mathbb{Q}$ such that $a < b$ and $|Y| = |\mathbb{Q}|$. Then $\text{rank}(T(\mathbb{Q}, Y) : O(\mathbb{Q}, Y)) = 1$.

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THANK YOU FOR YOUR ATTENTION.