Quasivariety of pseudo BCI-algebras and its properties

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A pseudo BCI-algebra is an algebra \((A, \rightarrow, \leadsto, 1)\), where \(\rightarrow\) and \(\leadsto\) are binary operations on \(A\) and 1 is an element of \(A\), satisfying the following axioms, for all \(x, y, z \in A\):

((A1)) \((x \rightarrow y) \leadsto ((y \rightarrow z) \leadsto (x \rightarrow z)) = 1\),

((A2)) \((x \leadsto y) \rightarrow ((y \leadsto z) \rightarrow (x \leadsto z)) = 1\),

((A3)) \(1 \rightarrow x = x\),

((A4)) \(1 \leadsto x = x\),

((A5)) if \(x \rightarrow y = 1\) and \(y \rightarrow x = 1\) then \(x = y\).

(W. A. Dudek, Y. B. Yun, 2008)

- The relation \(\leq = \{(x, y) \in A^2 \mid x \rightarrow y = 1\}\) is a partial order on \(A\) with 1 as a maximal element.
- If 1 is the greatest element of \(A\) then \((A, \rightarrow, \leadsto, 1)\) is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations \(\rightarrow\) and \(\leadsto\) coincide then \((A, \rightarrow, 1)\) is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety (relatively 1-regular, relatively ideal determined, relatively congruence modular (arguesian), 1-conservative).
A pseudo BCI-algebra is an algebra \((A, \rightarrow, \sim\rightarrow, 1)\), where \(\rightarrow\) and \(\sim\rightarrow\) are binary operations on \(A\) and \(1\) is an element of \(A\), satisfying the following axioms, for all \(x, y, z \in A\):

(A1) \((x \rightarrow y) \sim\rightarrow ((y \rightarrow z) \sim\rightarrow (x \rightarrow z)) = 1\),

(A2) \((x \sim\rightarrow y) \rightarrow ((y \sim\rightarrow z) \rightarrow (x \sim\rightarrow z)) = 1\),

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A pseudo BCI-algebra is an algebra \((A, \to, \rightsquigarrow, 1)\), where \(\to\) and \(\rightsquigarrow\) are binary operations on \(A\) and 1 is an element of \(A\), satisfying the following axioms, for all \(x, y, z \in A\):

\[(A1) \quad (x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1,\]
\[(A2) \quad (x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1,\]
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Embeding into the \{→, ⇔, 1\}-reduct of residuated po-monoid

- Every pseudo-BCI-algebra is a \{→, ⇔, 1\}-subreduct of an semi-integral residuated po-monoid.

**Semi-integral residuated po-monoid:**  
\((M, \leq, \cdot, →, ⇔, 1)\), where \((M, \cdot, 1)\) is a monoid, \(\leq\) is a partial order on \(M\), and \(→, ⇔\) are binary operations on \(M\) satisfying the *residuation law*, for all \(x, y, z \in M\):

\[
x \leq y \rightarrow z \quad \text{iff} \quad x \cdot y \leq z,
\]

\[
x \leq y \leftrightarrow z \quad \text{iff} \quad y \cdot x \leq z.
\]

The monoid identity \(1\) is a maximal element of the poset \((M, \leq)\).
In any pseudo BCI-algebra \((A, \to, \bowtie, 1)\) hold for all \(x, y, z \in A\):

1. \(x \to x = 1, x \bowtie x = 1\),
2. \(x \bowtie ((x \to y) \bowtie y) = 1, x \to ((x \bowtie y) \to y) = 1\),
3. \(x \to y = 1 \iff x \bowtie y = 1\),
4. \(x \leq y \implies y \to z \leq x \to z \text{ and } y \bowtie z \leq x \bowtie z\),
5. \(x \leq y \implies z \to x \leq z \to y \text{ and } z \bowtie x \leq z \bowtie y\),
6. \(x \to (y \bowtie z) = y \bowtie (x \to z)\),
7. \(x \leq y \to z \iff y \leq x \bowtie z\),
8. \(x \to y \leq (y \to x) \to 1, x \bowtie y \leq (y \bowtie x) \bowtie 1\),
9. \(x \to 1 = x \bowtie 1\),
10. \((x \to y) \to 1 = (x \to 1) \bowtie (y \to 1), (x \bowtie y) \bowtie 1 = (x \bowtie 1) \to (y \bowtie 1),\)
11. \(((x \to y) \bowtie y) \to y = x \to y\), \(((x \bowtie y) \to y) \bowtie y = x \bowtie y\),
12. \((x \to y) \to ((z \to x) \to (z \to y)) = 1, (x \bowtie y) \bowtie ((z \bowtie x) \bowtie (z \bowtie y)) = 1,\)
Important subalgebras of pseudo-BCI-algebra \((A, \to, \leadsto, 1)\):

- **Integral part of** \(A\) ... \(I_A = \{a \in A | a \leq 1\}\) - 1 is the top element of \(I_A\), i.e. \(I_A\) is a pseudo BCK-algebra.

\[ x \in I_A \iff ((x \to 1) \to 1) \to x = x \]

- **Group part of** \(A\) ... \(G_A = \{a \to 1 | a \in A\}\)

\[ x \in G_A \iff ((x \to 1) \to 1) = x \]

**Theorem**

\((G_A, \cdot)\) where \(x \cdot y = (x \to 1) \leadsto y = (y \leadsto 1) \to x\) is a group in which 1 is the identity and \(x^{-1} = x \to 1 = x \leadsto 1\) is the inverse of \(x \in G_A\). The original operations \(\to\) and \(\leadsto\) on \(G_A\) are retrieved from \(\cdot\) by \(x \to y = y \cdot x^{-1}\) and \(x \leadsto y = x^{-1} \cdot y\).

**Proposition**

For every \(a \in A\), \((a \to 1) \to 1\) is the only element \(g \in G_A\) with \(a \leq g\) and \(G_A\) is the set of all maximal elements of \(A\).
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For every \(a \in A\), \((a \rightarrow 1) \rightarrow 1\) is the only element \(g \in G_A\) with \(a \leq g\) and \(G_A\) is the set of all maximal elements of \(A\).
Example:
The set $A = \{0, 1, 2, 3, 4, 5\}$ equipped with the operations $\rightarrow$ and $\sim\rightarrow$ given by the following tables is a proper pseudo BCI-algebra:

<table>
<thead>
<tr>
<th>$\rightarrow$</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>1</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
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<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
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</tbody>
</table>

<table>
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<tr>
<th>$\sim\rightarrow$</th>
<th>0</th>
<th>2</th>
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</tr>
</tbody>
</table>
For the pseudo BCI-algebra we have $I_A = \{0, 1, 2\}$ and $G_A = \{1, 3, 4\}$ with the group operation $\cdot$ given by the following table:

<table>
<thead>
<tr>
<th></th>
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<th>3</th>
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<tbody>
<tr>
<td>1</td>
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Given a pseudo-BCI-algebra $A = (A, \rightarrow, \rightsquigarrow, 1)$, the algebra $A^* = (A, \rightsquigarrow, \rightarrow, 1)$ is a pseudo-BCI-algebra, too. $A$ and $A^*$ have the same underlying poset $(A, \leq)$, but it can easily happen that the algebras $A$ and $A^*$ are not isomorphic. For example, the “prelinearity identities”

$$(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$$

and

$$(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$$

are independent in general.

Group part $G_A^* = (G_A, \rightsquigarrow, \rightarrow, 1)$:

$$(G_A, \star), \text{ where } g \star h = h \cdot g \text{ for all } g, h \in G_A, \text{ is a group isomorphic to } (G_A, \cdot) \text{ (isomorphism } g \mapsto g^{-1}).$$

$G_A^* = (G_A, \rightsquigarrow, \rightarrow, 1) \ldots g \rightsquigarrow h = h \star g^{-1}, \ g \rightarrow h = g^{-1} \star h$
Relative congruences, filters and prefilters

Let $\mathcal{K}$ be a class of algebras of type $F$, $A \in \mathcal{K}$ and $\theta \in \text{Con}(A)$.
We say that $\theta$ is a **relative congruence** (or $\mathcal{K}$-congruence) on $A$ if $A/\theta \in \mathcal{K}$.

Denote $\text{Con}_{\mathcal{K}}(A)$ the set of all relative congruences on $A$.

A **prefilter** in a pseudo BCI-algebra $A$ is $D \subseteq A$ such that

(i) $1 \in D$,  
(ii) if $x \in D$ and $x \rightarrow y \in D$, then $y \in D$,  
(iii) for all $x \in A$, if $x \in D$ then $x \rightarrow 1 \in D$.

A prefilter $D$ is a **filter** if, for all $x, y \in A$,

$$x \rightarrow y \in D \quad \text{iff} \quad x \Rightarrow y \in D.$$ 

Denote $\mathcal{F}(A)$ the set of all filters on $A$.  
$I_A$ is always a filter of $A$. 

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Pseudo BCI-algebras
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Definition of pseudo BCI-algebras
Properties of pseudo BCI-algebras
Prefilters and filters of pseudo BCI-algebras
Properties of quasivariety of pseudo BCI-algebras
The End
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Denote $\mathcal{F}(A)$ the set of all filters on $A$. $I_A$ is always a filter of $A$. 

Lemma
Let \((A, \to, \rightsquigarrow, 1)\) be a pseudo BCI-algebra.

1. Any prefiltr is an order-filter, i.e., \(x \in D\) and \(x \leq y\) imply \(y \in D\).

2. Any prefilter is a subalgebra of \((A, \to, \rightsquigarrow, 1)\).

3. \(D \subseteq A\) is a prefilter if and only if \(1 \in D\) and

\(\text{(ii')}\) for all \(x, y \in A\), if \(x \rightsquigarrow y \in D\) and \(x \in D\) then \(y \in D\).

\(\text{(iii')}\) for all \(x \in A\), if \(x \in D\) then \(x \rightsquigarrow 1 \in D\).

The filters correspond to the relative congruences: for every filter \(D\),

\[\theta_D = \{(x, y) \mid x \to y \in D\text{ and } y \to x \in D\}\]

is the only relative congruence such that \([1]_{\theta_D} = D\).
Lemma
Let \( (A, \to, \multimap, 1) \) be a pseudo BCI-algebra.

1. Any prefiltr is an order-filter, i.e., \( x \in D \) and \( x \leq y \) imply \( y \in D \).
2. Any prefilter is a subalgebra of \( (A, \to, \multimap, 1) \).
3. \( D \subseteq A \) is a prefilter if and only if \( 1 \in D \) and

(ii’) for all \( x, y \in A \), if \( x \multimap y \in D \) and \( x \in D \) then \( y \in D \).

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Remark
It is easy to show that

$$\theta = \{(x, y) \mid x \rightarrow 1 = y \rightarrow 1\}$$

is a congruence on $A$ such that $[1]_{\theta} = I_A$, i.e. $\theta = \theta I_A$.

Remark
The map $\alpha : a \mapsto a \rightarrow 1 = a \rightsquigarrow 1$ is a homomorphism of $(A, \rightarrow, \rightsquigarrow, 1)$ onto $(G_A, \rightsquigarrow, \rightarrow, 1)$ with kernel congruence $\theta$, i.e. $(A, \rightarrow, \rightsquigarrow, 1)/\theta \cong (G_A, \rightsquigarrow, \rightarrow, 1)$.

The map $\beta : a \mapsto (a \rightarrow 1) \rightarrow 1$ is a homomorphism of $(A, \rightarrow, \rightsquigarrow, 1)$ onto $(G_A, \rightarrow, \rightsquigarrow, 1)$ with kernel congruence $\theta$, i.e. $(A, \rightarrow, \rightsquigarrow, 1)/\theta \cong (G_A, \rightarrow, \rightsquigarrow, 1)$.

$x \rightarrow 1 = y \rightarrow 1$ iff $(x \rightarrow 1) \rightarrow 1 = (y \rightarrow 1) \rightarrow 1$ iff

$x \rightarrow y, y \rightarrow x \in I_A$
Remark

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$$x \to 1 = y \to 1 \text{ iff } (x \to 1) \to 1 = (y \to 1) \to 1 \text{ iff } x \to y, y \to x \in I_A$$
Theorem
Let \((G, \cdot)\) be a group with the identity \(e\) and define \(x \rightarrow y = y \cdot x^{-1}\) and \(x \bowtie y = x^{-1} \cdot y\). Then \((G, \rightarrow, \bowtie, e)\) is a trivially ordered pseudo BCI-algebra where \(\emptyset \neq H \subseteq G\) is a prefilter iff it is a subgroup of \((G, \cdot)\) and \(H\) is a filter iff it is a normal subgroup of \((G, \cdot)\).

Corollary
The lattice of prefilters need not be modular. The lattice of filters need not be distributive.
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\[ x \leadsto y = x^{-1} \cdot y. \] Then \((G, \rightarrow, \leadsto, e)\) is a trivially ordered pseudo BCI-algebra where \(\emptyset \neq H \subseteq G\) is a prefilter iff it is a subgroup of \((G, \cdot)\) and \(H\) is a filter iff it is a normal subgroup of \((G, \cdot)\).

Corollary

The lattice of prefilters need not be modular. The lattice of filters need not be distributive.
The direct product of $I_A$ and $G_A$

For any $a, b \in A$ we define

\[ x \circ y = (x \to 1) \bowtie y, \]
\[ x \star y = (y \bowtie 1) \to x. \]

For the operations $\circ$ and $\star$ we have

1. $1 \circ x = x$, $x \star 1 = x$,
2. $x \circ (x \to 1) = 1$, $(x \bowtie 1) \star x = 1$,
3. $x \circ (y \star z) = (x \circ y) \star z$,
4. $(x \circ y) \circ z \leq x \circ (y \circ z)$,
5. $x \star (y \star z) \leq (x \star y) \star z$. 
The direct product of $I_A$ and $G_A$

For any $a, b \in A$ we define
\[ x \circ y = (x \rightarrow 1) \leadsto y, \]
\[ x \star y = (y \leadsto 1) \rightarrow x. \]

For the operations $\circ$ and $\star$ we have
1. $1 \circ x = x$, $x \star 1 = x$,
2. $x \circ (x \rightarrow 1) = 1$, $(x \leadsto 1) \star x = 1$,
3. $x \circ (y \star z) = (x \circ y) \star z$,
4. $(x \circ y) \circ z \leq x \circ (y \circ z)$,
5. $x \star (y \star z) \leq (x \star y) \star z$. 
Lemma

For any pseudo-BCI-algebra $A$, the following are equivalent:

1. The operation $\circ$ is associative;
2. $(g \circ h) \circ x = g \circ (h \circ x)$ for all $g, h \in G_A$ and $x \in A$;
3. $g^{-1} \rightsquigarrow (g \rightsquigarrow x) = x$ for all $g \in G_A$ and $x \in A$;
4. The operation $\star$ is associative;
5. $x \star (h \star g) = (x \star h) \star g$ for all $g, h \in G_A$ and $x \in A$;
6. $g^{-1} \rightarrow (g \rightarrow x) = x$ for all $g \in G_A$ and $x \in A$. 
Lemma

Let $A$ be a pseudo-$BCI$-algebra. Then for all $x \in A$ and $g, h \in G_A$:

(i) $x \rightarrow g = (g \rightarrow x) \rightarrow 1$, $x \bowtie g = (g \bowtie x) \rightarrow 1$;

(ii) $g \rightarrow x = x \star g^{-1}$, $g \bowtie x = g^{-1} \circ x$;

(iii) $(x \rightarrow g) \rightarrow 1 = x \circ g^{-1}$, $(x \bowtie g) \rightarrow 1 = g^{-1} \star x$.

Proof.

We have $(g \rightarrow x) \rightarrow 1 = (g \rightarrow 1) \bowtie (x \rightarrow 1) = x \rightarrow ((g \rightarrow 1) \bowtie 1) = x \rightarrow g$, and similarly, $(g \bowtie x) \rightarrow 1 = x \bowtie g$.

Clearly, $x \star g^{-1} = (g^{-1} \bowtie 1) \rightarrow x = g \rightarrow x$,

$g^{-1} \circ x = (g^{-1} \rightarrow 1) \bowtie x = g \bowtie x$,

$x \circ g^{-1} = (x \rightarrow 1) \bowtie (g \rightarrow 1) = (x \rightarrow g) \rightarrow 1$ and

$g^{-1} \star x = (x \bowtie 1) \rightarrow (g \bowtie 1) = (x \bowtie g) \rightarrow 1$.  

\[\square\]
Theorem

Let $A$ be a pseudo-BCI-algebra. The following statements are equivalent:

(1) $A \simeq I_A \times G_A \simeq I_A \times G_A^*$;

(2) $G_A$ is a filter of $A$;

(3) $A$ satisfies the equivalent conditions (1) - (6) of the Lemma and

\[ g \to x = g \bowtie x \quad \text{for all } g \in G_A, \ x \in I_A. \]
Properties of quasivariety of pseudo BCI-algebras

1) Quasivariety of pseudo BCI-algebras is relatively regular in 1

A quasi-variety $\mathcal{H}$ with a constant term 1 is said to be relatively regular in 1, if $[1]_\theta = [1]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \text{Con}_\mathcal{H}(A)$.

It is known that a quasi-variety $\mathcal{H}$ is relatively regular in 1 iff there exist the terms $d_1(x, y), \ldots, d_n(x, y)$ in $\mathcal{H}$ such that $d_1(x, y) = 1, \ldots, d_n(x, y) = 1$ implies $x = y$.

Obviously, for the quasi-variety of all pseudo-BCI-algebras we can take $n = 2$, $d_1(x, y) = x \rightarrow y$ and $d_2(x, y) = y \rightarrow x$. 
2) Quasivariety of pseudo BCI-algebras is relatively ideal determined

Let $\mathcal{K}$ be a class of algebras of type $F$ with a constant 1. A term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ of type $F$ is called an **ideal term** in $y_1, \ldots, y_n$ if $\mathcal{K} \models t(x_1, \ldots, x_m, 1, \ldots, 1) = 1$.

A non-empty subset $I$ of $A$ is said to be **closed under the ideal term** $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in $y_1, \ldots, y_n$ if $t(a_1, \ldots, a_m, b_1, \ldots, b_n) \in I$ whenever $b_1, \ldots, b_n \in I$.

We say that $\emptyset \neq I \subseteq A$ is an **ideal** in $A$ if it is closed under all ideal terms.

The class $\mathcal{K}$ is called **relatively ideal determined** if for each $A \in \mathcal{K}$, every ideal in $A$ is the kernel of a unique relative congruence on $A$.

Denote $\mathcal{I}(A)$ the set of all ideals on $A$. 

Theorem
Let $A$ be a pseudo BCI-algebra and $I \subseteq A$ with $1 \in I$. The following statements are equivalent:

(i) $I$ is a filter of $A$.
(ii) $I = [1]_\theta$ for some $\theta \in \text{Con}_K(A)$.
(iii) $I$ is an ideal of $A$.
(iv) $I$ is closed with respect to the ideal terms

$$t_1(x_1, x_2, y_1, y_2) = (((y_1 \rightarrow (y_2 \rightarrow x_1)) \rightarrow x_1) \sim x_2) \sim x_2$$

$$t_2(y) = y \rightarrow 1$$

(v) $I$ is closed with respect to the ideal terms

$$w_1(x, y_1, y_2) = (y_1 \rightarrow (y_2 \rightarrow x)) \rightarrow x$$

$$w_2(x, y) = (y \sim x) \sim x$$

$$t_2(y) = y \rightarrow 1$$
Corollary

\[ \text{Con}_\mathcal{K}(A) \cong \mathcal{I}(A) = \mathcal{F}(A). \]

Let \( \mathcal{K} \) be a relatively regular in 1 quasivarity in which there is a one-one correspondence between ideals and relative congruences, that is, for every algebra \( A \in \mathcal{K} \), the map \( \theta \mapsto [1]_\theta \) is an isomorphism of \( \text{Con}_\mathcal{K}(A) \) onto \( \mathcal{I}(A) \). Then the following lemma holds:

Lemma

Let \( \alpha, \beta \in \text{Con}_\mathcal{K}(A) \). Then

\[ [1]_\alpha \lor [1]_\beta = \{ a \in A \mid (a, b) \in \alpha \text{ for some } b \in [1]_\beta \} = \{ a \in A \mid (a, b) \in \beta \text{ for some } b \in [1]_\alpha \}, \]

i.e. \( a \in [1]_\alpha \lor [1]_\beta \) iff \( (a, 1) \in \alpha \circ \beta \) iff \( (a, 1) \in \beta \circ \alpha \).

Corollary

The lattices \( \text{Con}_\mathcal{K}(A) \cong \mathcal{I}(A) = \mathcal{F}(A) \) are modular.
3) Quasivariety of pseudo BCI-algebras is 1-conservative

A quasivariety $Q$ with a constant 1 is said to be **1-conservative** if $Q$ and the variety $\text{HSP}(Q)$ generated by $Q$ satisfy the same quasi-identities of the form

$$\bigwedge_{i=1}^{n} s_i(x_1, \ldots, x_k) = 1 \implies t(x_1, \ldots, x_k) = 1.$$ 

**Proposition**

Let $Q$ be a relatively 1-regular quasivariety. Then $Q$ is relatively ideal determined if and only if $Q$ is 1-conservative and has a subtractive term (a binary term $s(x, y)$ such that $Q$ satisfies the identities $s(x, x) = 1$ and $s(x, 1) = x$).

**Corollary**

Quasivariety of pseudo BCI-algebras is 1-conservative (subtractive term $s(x, y) = y \rightarrow x$).


THANK YOU FOR YOUR ATTENTION!