On Finite Generated Subsemigroups of $T(X, Y)$

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- $rank(S : R) := \min\{|A| : A \subseteq S, \langle R \cup A \rangle = S\}$ **relative rank** of S **modulo** R

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- If $rank(T_X : R)$ is countable then there is countable set $Q \subseteq T_X$ with $T_X = \langle R \cup Q \rangle$
- there are $\alpha, \beta \in T_X$ with $Q \subseteq \langle \alpha, \beta \rangle$
- $T_X = \langle R, \alpha, \beta \rangle$, i.e. $rank(T_X : R) \leq 2$.

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Sierpinski´s Theorem for $T_X(Y)$

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- We have $\theta_n = \beta^2 \alpha \beta^n \alpha^2$.

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Lemma

Let $\theta_1, \ldots, \theta_n, \ldots \in T_X(Y)$ countable and $Y \subseteq X$ finite subset of an infinite set X. Then rank$(\langle \theta_1, \ldots, \theta_n, \ldots \rangle) = \aleph_0$.

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- *iii)* Y *is not finite or there is partition* $\{A_y : y \in Y\}$ *of* X *s.t.* $\forall y \in Y$ *and* $\forall n \in \mathbb{N} \exists z \in Y$ *with* $A_y \subseteq z\theta_n^{-1}$.

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- $\theta_n = \alpha\beta$ for some $\alpha \in A$ and $\beta \in T_X(Y)$

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It depends from the concrete choose of the countable set Q.

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- Let $A_1 := D(\mathbb{Q})$
- Let $A_{i+1} := \bigcup\{D(Y) \mid Y \in A_i\}$ for $i \in \mathbb{N}$

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- For $Y \subseteq \mathbb{Q}$ with $|Y| = \aleph_0$ we choose a decomposition $D(Y) = \{Y_i \mid i \in \mathbb{N}\}$ with $|Y_i| = \aleph_0$
- For each decomposition D of $\mathbb{Q}$, we choose a surjective map $\beta_D : \mathbb{Q} \to \mathbb{N}$ with $\ker \beta_D = D$
- Let $A_1 := D(\mathbb{Q})$
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- $\theta_i := \beta_{A_i} \in T_{\mathbb{Q}}(\mathbb{N})$ for $i \in \mathbb{N}$

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Fact

$rank(\langle \theta_1, \ldots, \theta_n, \ldots \rangle) = \aleph_0$