Increasing the role of the *D*-basis in applications

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Novi Sad, Serbia

Based on papers:

“Ordered direct basis of a finite closure system"
joint work with J.B.Nation and R.Rand, Disc. Appl. Math. 2013

“Discovery of the D-basis in binary tables based on hypergraph dualization", 2012
joint work with J.B.Nation, submitted to Theoretical Computer Science, in arxiv

“Measuring the implications of the D-basis in analysis of data in biomedical studies", 2015
joint work with J.B.Nation, G. Okimoto, K. Alibek and others, Proceedings of ICFCA-2015, Spain

Last paper's support:

The project has been supported by the research grant N 13/42
" Algebraic methods of data retrieval"
Nazarbayev University, 2013-2015
and grant N 0112PK02175, 2012-2014,
Ministry of Healthcare and Social Development of RK

History of events

- Novi Sad Algebraic Conference, 2005, Kira's talk on "Relatively convex sets and Jamison problem"
- meeting Gyuri Turan, AMS conference in Urbana-Champaign in 2009
- visiting Turan and Bob Sloan in Chicago in 2010, Bertet-Monjardet's paper in TCS, 2010, just out
- first draft of D-basis paper, summer 2010 in Chicago
- Robert Rand and his honor's project, 2010-2011
- talk on the D-basis at RUTCOR seminar, with Endre Boros and Vladimir Gurvich attending, October 2011
- visit of Karell Bertet to New York, May 2012
- first version of paper on the D-basis retrieval, November 2012
- Kira's arrival in Astana, on-line course with Yeshiva University students, grant of Nazarbayev University, Spring 2013

Outline

Closure systems

$\langle X, \phi \rangle$ is *a closure system*, if

- X is non-empty set (finite in this talk);
- ϕ is a closure operator on X, i.e. $\phi : 2^X \rightarrow 2^X$ with
  (1) $Y \subseteq \phi(Y)$;
  (2) $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$;
  (3) $\phi(\phi(Y)) = \phi(Y)$, for all $Y, Z \subseteq X$.
- Closed set: $A = \phi(A)$;
- Lattice of closed sets: $Cl(X, \phi)$.

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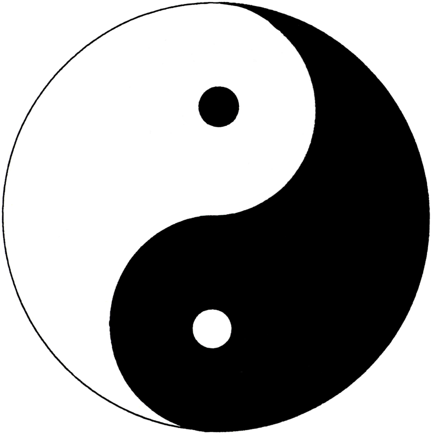
Yin Yang of a closure system

Algebraic part of a closure system: lattice of closed sets



Logic part of a closure system: set of implications.

Lattices and closure systems

Proposition

Every finite lattice L is the lattice of closed sets of some closure system $\langle X, \phi \rangle$.

- Take $X = \mathrm{Ji}(L)$, the set of join-irreducible elements: $j \in \mathrm{Ji}(L)$, if $j \neq 0$, and $j = a \vee b$ implies $j = a$ or $j = b$;
- Define $\phi(Y) = \{j \in \mathrm{Ji}(L) : j \leq \bigvee Y\}$, $Y \subseteq X$.
- Proof of $L \cong Cl(X, \phi)$: every element $x \in L$ corresponds to ϕ-closed set of all join irreducibles below x.
- So defined closure system is always *standard*.

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Standard closure systems

- Closure system $\langle X, \phi \rangle$ is standard, when for no $x \in X$ there exists $Y \subseteq X \setminus \{x\}$ such that $\phi(x) = \phi(Y)$.
- For every closure system $\langle Y, \phi \rangle$ one can find $X \subseteq Y$ such that, with restriction ϕ_X of ϕ on X, one obtains the *standard* closure system $\langle X, \phi_X \rangle$, with the lattice of closed sets isomorphic to $Cl(Y, \phi)$.
- Moreover, for every $y \in Y \setminus X$ we have $\phi(y) = \phi(X')$, for some $X' \subseteq X$.

Example: Building a closure system associated with lattice A_{12}.
$X = \mathrm{Ji}(A_{12}) = \{1, 2, 3, 4, 5, 6\}$. $\phi(\{4, 6\}) = \{1, 3, 4, 6\}$, $\phi(\{2, 4\}) = X$ etc.

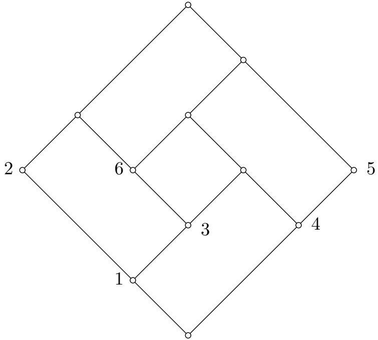

Figure: A_{12}

Closure systems and implications

- An implication σ on X: $Y \to Z$, for $Y, Z \subseteq X$, $Z \neq \emptyset$.
- σ-closed subset A of X: if $Y \subseteq A$, then $Z \subseteq A$.
- Closure system $\langle X, \phi_S \rangle$ defined by set $\mathcal{S}$ of implications on X: A is closed, if it is σ-closed, for each $\sigma \in \mathcal{S}$
- Every closure system $\langle X, \psi \rangle$ can be presented as $\langle X, \phi_S \rangle$, for some set $\mathcal{S}$ of implications on X.
- Example: $\mathcal{S} = \{A \to \phi(A) : A \subseteq X, A \neq \phi(A)\}$.

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Operator and sets of implications

Note:

- Every set of implications $\mathcal{S}$ on X defines *unique* closure operator on X.
- There exist *numerous* sets of implications that define the same operator on X.

Example: Let $X = \{a, b, c\}$. Consider $\mathcal{S}_1 = \{a \to bc\}$ and $\mathcal{S}_2 = \{a \to bc, ab \to c, ac \to b, a \to b, bc \to c\}$.
The closure systems defined by $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same.

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The bases of a closure system

Term *a base* or *a basis* is used when the set of implications $\mathcal{S}'$ that defines the same closure system satisfies some condition of minimality.

We mention two famous bases: canonical and canonical unit direct (CUD).

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J.L. Guiques, V. Duquenne, *Familles minimales d'implications informatives résultant d'une tables de données binares*, Math. Sci. Hum. **95** (1986), 5–18.

- Define *critical subsets of* X for a given closure system $\langle X, \phi \rangle$.
- Canonical basis S_C is $\{C \to B : C$ is critical, $B = \phi(C) \setminus C\}$.
- If S is any other set of implications generating $\langle X, \phi \rangle$, then for every critical set C one can find $(C' \to D) \in S$ such that $C' \subseteq C$, and no other critical or closed set Y with $C' \subseteq Y \subset C$.
- S_C is the *minimum* basis among all the bases generating $\langle X, \phi \rangle$.

Definition

Set of implications S defining ϕ on A is called minimum basis if $|S|$ is minimal among all sets of implications defining ϕ.

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For every set of implications $\mathcal{S}$, one can find the canonical basis $\mathcal{S}_C$ defining the same operator in time $O(|s(\mathcal{S})|^2)$.

Here the size $s(\mathcal{S})$ of the set of implications $\mathcal{S} = \{X_i \to Y_i : i \leq n\}$, is the number $s(\mathcal{S}) = |X_1| + |Y_1| + \cdots + |X_n| + |Y_n|$.

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Canonical direct unit basis

K.Bertet, B.Monjardet, *The multiple facets of the canonical direct unit implicational basis*, Theoretical Computer Science 411 (2010), 2155-2166.

Basis $\mathcal{S}$ is *unit*, if it comprises implications $Y \to b$, with the singleton $b \in X$ on the right.

Given unit basis $\mathcal{S}$ and $Y \subset X$, define
$\pi_{\mathcal{S}}(Y) = Y \cup \bigcup\{b : (A \to b) \in \mathcal{S}, A \subseteq Y\}$.
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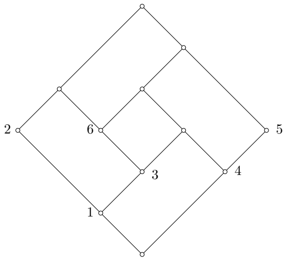
$\phi_{\mathcal{S}}(Y) = \pi_{\mathcal{S}}(Y)$, for all $Y \subseteq X$.

Example

Canonical basis $\mathcal{S}_C$ of $\langle \mathrm{Ji}(A_{12}), \phi \rangle$ has 8 implications:
$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5.$
Consider $Y = \{2,4\}$. Then $\pi(Y) = \{2,4,1\}$, $\pi^2(Y) = \{2,4,1,3\}$, $\pi^3(Y) = \{2,4,1,3,6\}$, $\pi^4(Y) = \{1,2,3,4,5,6\} = \phi(Y)$. This basis is *not* direct.



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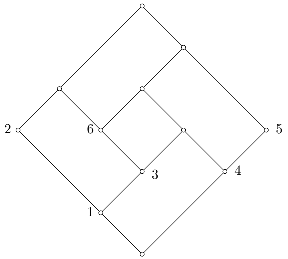

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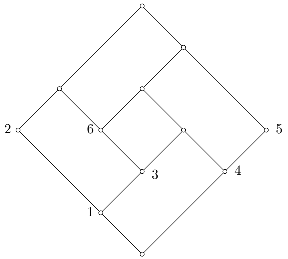

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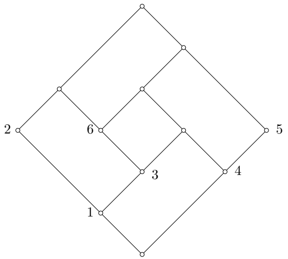

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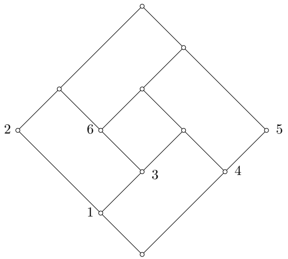

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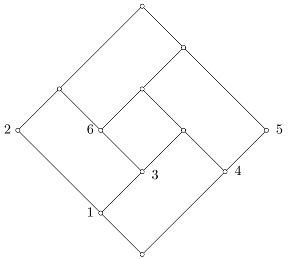

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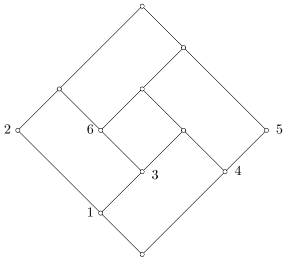
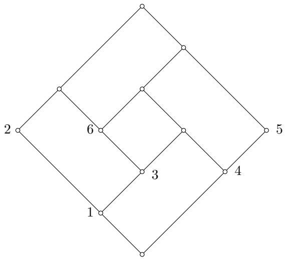


Example: continued

Canonical unit basis $\mathcal{S}_U$ for $\langle \mathrm{Ji}(A_{12}), \phi \rangle$ has 13 implications:
$2 \to 1, 6 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 24 \to 3, 15 \to 3,$
$23 \to 6, 15 \to 6, 25 \to 6, 24 \to 5, 24 \to 6.$

Consider $Y = \{2, 4\}$. Then $\pi(Y) = \{2, 4, 1, 3, 5, 6\} = \phi(Y)$. This basis is direct.



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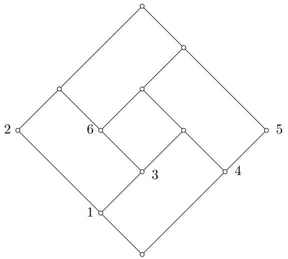

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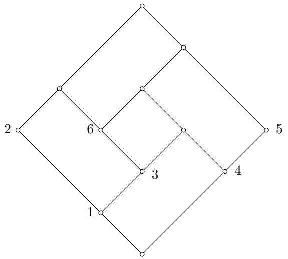

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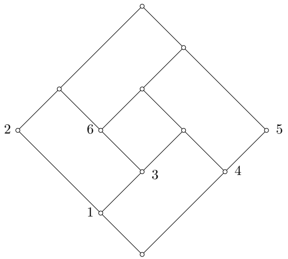

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OD-graph of a finite lattice:
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The full information about finite lattice L can be compactly recorded in

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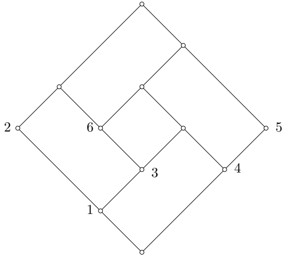

Figure: A_{12}

For lattice A_{12}, the poset of join-irreducible elements is:
$\langle \mathrm{Ji}(A_{12}), \leq \rangle = \langle \{1, 2, 3, 4, 5, 6, \}, 1 \leq 2, 1 \leq 3 \leq 6, 4 \leq 5 \rangle$.

Example continued

A *join-cover* is an expression $j \leq j_1 \vee \cdots \vee j_k$, $j \nleq j_i$, $i \leq k$, for some $j, j_1, \ldots, j_k \in \mathrm{Ji}(L)$.

Examples: $3 \leq 1 \vee 4$, or $6 \leq 2 \vee 5$.

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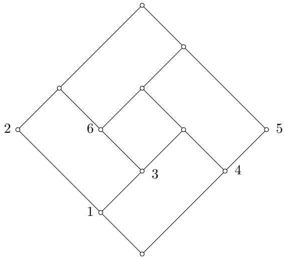

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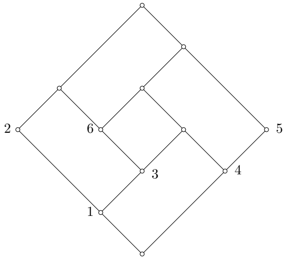

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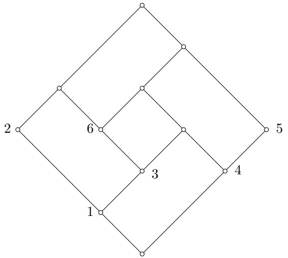

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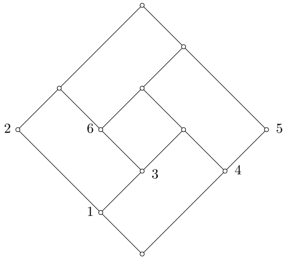

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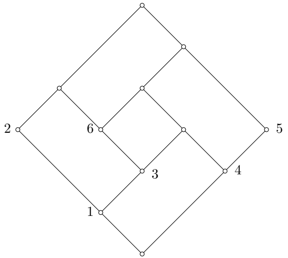

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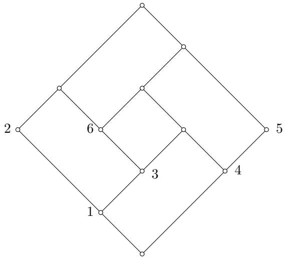

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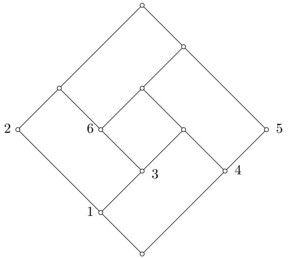

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Definition

Let $\langle X, \phi \rangle$ be a standard closure system with $L = Cl(X, \phi)$. The set of implications $\mathcal{S}_D$ is called the D-basis of $\langle X, \phi \rangle$, if it is made of two parts:

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Let $\langle X, \phi \rangle$ be a standard closure system with $L = Cl(X, \phi)$.
The set of implications $\mathcal{S}_D$ is called the D-basis of $\langle X, \phi \rangle$, if it is made of two parts:

- $\{a \to b : b \in \phi(a)\}$*; equivalently,* $b \leq a$ *in* $\langle Ji(L), \leq \rangle$*.*
  This part is called a binary part of the basis.
- $\{j_1 \ldots j_k \to j : j \leq j_1 \vee \cdots \vee j_k$ *is a minimal cover in* $L\}$*.*

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The *D*-relation and the *D*-basis

Why *D* in the name of the basis?

D-relation is an important concept in the study of free lattices, see R. Freese, J. Jezek, J.B. Nation “Free Lattices", 1995.

Definition

Given $b, c \in \mathrm{Ji}(L)$*, one defines bDc, when there is a minimal cover* $b \leq c \vee j_1 \vee \dots \vee j_k$*, for some* $j_1, \dots, j_k \in \mathrm{Ji}(L)$*.*

Equivalently: bDc iff there exists $Y \to b$ in the *D*-basis such that $c \in Y$.

Important: for every $Y \to b$ in the *D*-basis, $Y \subseteq bD = \{c \in \mathrm{Ji}(L) : bDc\}$.

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The *D*-basis and the canonical unit basis

Theorem (ANR-2013)

- S_D *generates* $\langle X, \phi \rangle$*, i.e., D-basis is, indeed, a basis of this closure system.*
- S_D *is a subset of the canonical unit basis* S_U*.*

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Comparison

Canonical direct unit basis $\mathcal{S}_U$ for $\langle \mathrm{Ji}(A_{12}), \phi \rangle$ has 13 implications.
$2 \to 1, 6 \to 1, 6 \to 3, 3 \to 1, 5 \to 4, 14 \to 3, 24 \to 3, 15 \to 3,$
$23 \to 6, 15 \to 6, 25 \to 6, 24 \to 5, 24 \to 6.$

D-basis has 9 implications.
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Ordered iteration

Suppose the set of implications $\mathcal{S}$ are put into some linear order:

$$\mathcal{S} = \langle s_1, s_2, \ldots, s_n \rangle.$$

A mapping $\rho_{\mathcal{S}} : 2^X \to 2^X$ associated with this ordering is called an *ordered iteration* of $\mathcal{S}$:

- For any $Y \subseteq X$, let $Y_0 = Y$.
- If Y_k is computed and implication s_{k+1} is $A \to b$, then

$$Y_{k+1} = \begin{cases} Y_k \cup \{b\}, & \text{if } A \subseteq Y_k, \\ Y_k, & \text{otherwise.} \end{cases}$$

- Finally, $\rho_{\mathcal{S}}(Y) = Y_n$.

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An implicational basis of $\langle X, \phi \rangle$, together with its order: $\mathcal{S} = \langle s_1, \ldots, s_n \rangle$ is called ordered direct, if $\rho(Y) = \phi(Y)$, for every $Y \subseteq X$.

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Example

Take $\mathcal{S}_C$, the set of implications for $\langle \mathrm{Ji}(A_{12}), \phi \rangle$, in its original order:
$2 \to 1, 6 \to 13, 3 \to 1, 5 \to 4, 14 \to 3, 123 \to 6, 1345 \to 6, 12346 \to 5$.

Consider $Y = \{2, 4\}$.

Then $\pi(Y) = \{2, 4, 1\}$, while
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Ordered direct basis

Theorem (ANR-2013)

- *$\mathcal{S}_D$ is an ordered direct basis, associated with any order, where the binary part precedes the rest of implications.*
- *There exist closure systems, for which the canonical basis cannot be ordered.*

Algorithmic aspects

If $\mathcal{S}$ is a any unit direct basis of $\langle X, \phi \rangle$ of size $s = s(\mathcal{S})$ with m implications, then

- it takes time $O(s^2)$ to extract D-basis from $\mathcal{S}$;
- it takes time $O(m)$ to put extracted D-basis into a proper order.

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Mid-talk conclusions

- For all practical purposes canonical direct unit basis can be replaced by the considerably shorter *D*-basis.
- The *D*-basis preserves the property of direct processing, assuming negligible pre-processing time for its ordering.

Binary tables and the Galois connection

	C_1	C_2	DE	PDE	MP
1	1	0	0	0	0
2	1	1	1	0	0
3	1	1	1	1	0
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$U = \{1, 2, 3, 4, 5, 6\}$ is the set of *objects*.

$A = \{C_1, C_2, DE, PDE, MP\}$ is the set of *attributes*.

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Closure systems associated with a binary table

Table $\mathcal{T} = \langle U, A, R \rangle$, where $R \subseteq A \times U$ is a *relation* between U and A.

$S_A : 2^A \to 2^U$ is a *support function* on A

$S_A(Z) = \{y \in U : (z, y) \in R, \text{ for all } z \in Z\}$, for $Z \in 2^A$.

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Lemma

Let $\mathcal{T} = \langle U, A, R \rangle$ be a table with support functions S_A and S_U.

- *S_A and S_U yield Galois connection between the power sets 2^A and 2^U.*
- *Mapping $\phi_A : Z \mapsto S_U(S_A(Z))$, with $Z \in 2^A$, is a closure operator on A.*
- *Similarly, mapping $\phi_U : Y \mapsto S_A(S_U(Y))$, with $Y \in 2^U$, is a closure operator on U.*

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Lattice and implicational sets of a binary table

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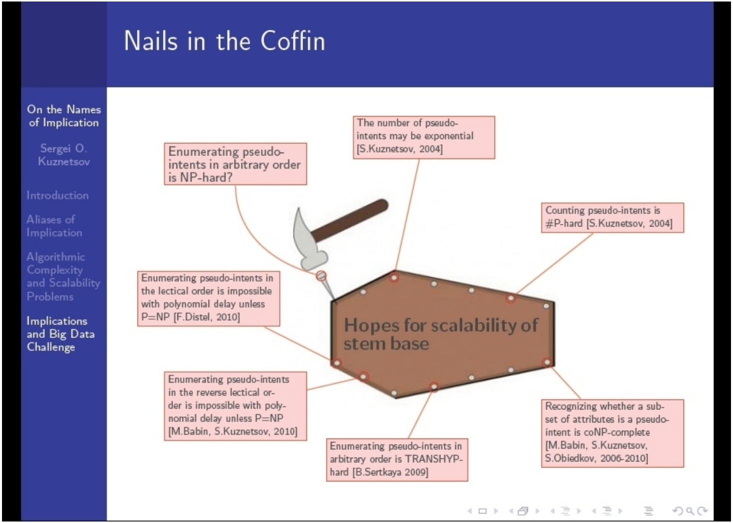
- The lattice of closed sets: *Galois lattice* or *concept lattice*.
- Implications: usually on the set of attributes.
  Examples: $(C_2 \to C_1)$, $(C_1, DE \to C_2)$.

Retrieval of a basis from a binary table

- As-of 2012 talk of K. Bertet at Combinatorics Seminar of CUNY, both problems of canonical and canonical unit direct bases generation from a binary table were reported open.
- All existing algorithms required generation of a closure system or a concept lattice, before attempting the basis retrieval.
- The size of the closure system or concept lattice is (worst case) exponential in the size of the table.

Complexity of retrieval of the canonical basis

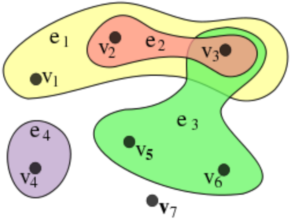
*Courtesy of Vincent Duquenne and Sergei Kuznetsov



Retrieval of the CUD basis and the *D*-basis

- U. Ryssel, F. Distel and D. Borchmann, *Fast algorithms for implication bases and attribute exploration using proper premises*, Ann. Math. Art. Intell. **70** (2014), 25–53.
- K. Adaricheva, J.B. Nation, *Discovery of the D-basis in binary tables based on hypergraph dualization*, arxiv, subm. TCS, 2015.

Hypergraph Dualization problem



- $V = \{v_1, \ldots, v_7\}$ is the set of vertices, $E = \{e_1, \ldots, e_4\} \subseteq \mathbf{2}^V$ is the set of hyper-edges.
- $H = \langle V, E \rangle$ is a hypergraph.
- $T \subseteq V$ is a *transversal*, if $T \cap e_i \neq \emptyset$, for all $e_i \in E$.
- Problem: find all *minimal* transversals of given hypergraph H.
- Solution: $H^d = \{V, E^d = \{v_4v_3, v_4v_2v_5, v_4v_2v_6\}\}$ is a dual hypergraph.

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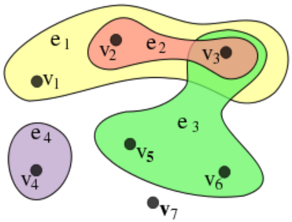

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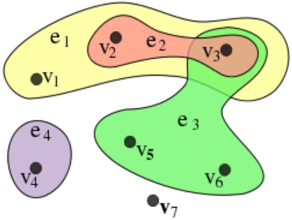

- $V = \{v_1, \ldots, v_7\}$ is the set of vertices, $E = \{e_1, \ldots, e_4\} \subseteq \mathbf{2}^V$ is the set of hyper-edges.
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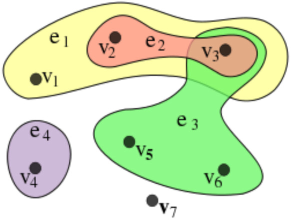

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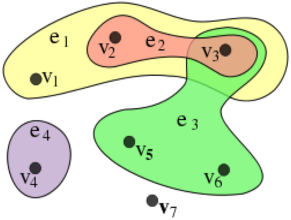

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Instance of HD problem for the *D*-basis retrieval

- Fix $b \in A$, one particular attribute. The goal: obtain all $Y \to b$ from the *D*-basis.
- Due to the definition of the *D*-basis, all such Y are subsets of $bD = \{c \in A : bDc\}$.
- Use Lemma 11.10 from *Free Lattices* book: bDc, for $b, c \in \mathrm{Ji}(L)$ iff there exists $p \in \mathrm{Mi}(L)$ such that $b \uparrow p$ and $p \downarrow c$.
- Attributes of the table play the role of join-irreducibles and the objects the role of meet-irreducibles of the concept lattice.
- $\uparrow$ and $\downarrow$ relations between the attributes and objects of the table can be found in time polynomial in the size of the table.
- Hypergraph associated with the fixed $b \in A$: set of vertices $V = bD$; hyperedges are $H_p = \{c \in bD : cRp\}$, for each $p \in U$, for which $b \uparrow p$.

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4th International Workshop "Algebra across the borders"

The workshop program will start on September 8-10, 2015, Tuesday to Thursday, in Nazarbayev University, in Astana, the new capital of Kazakhstan.

*Pictures: courtesy of J.B Nation

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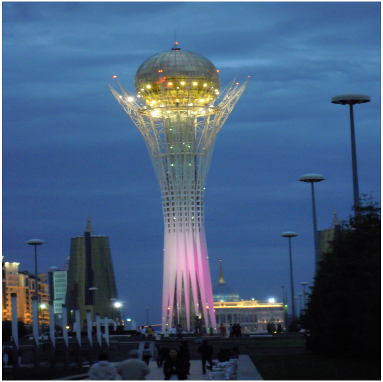

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Then we relocate to the Almaty region, for September 11-13, Friday to Sunday, for the second, less formal half of our program, consisting of additional lectures, mutual research collaboration, and opportunities for hiking in the mountains.

Contact: Kira Adaricheva or David Stanovsky

Regards from JB Nation

Figure: JB during hiking in NY State

Acknowledgments

The following people assisted in the project:

- Joshua Blumenkopf (Yeshiva College, New York, 3d year Physics major in 2013)
- Toviah Moldwin (Yeshiva College, New York, 3d year CS major in 2013)
- Takeaki Uno (National Institute of Informatics, Tokyo)
- Gordon Okimoto (University of Hawaii Cancer Center)
- Nazar Seidalin (hospital of Medical Holding, Astana)
- Kenneth Alibek (Graduate School of Medicine, NU)
- Vyacheslav Adarichev (Biology Department, NU)
- Adina Amanbekkyzy (Math Department TA, NU)
- Shuchismita Sarkar (Math Department TA, NU)
- Alibek Sailanbayev (2d year CS student, NU)
- Ulrich Norbisrath (CS Department, NU)
- Mark Sterling (CS Department, NU)