On the category of affine systems

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Outline

Topological systems

- In 1989, S. Vickers introduced the notion of *topological system* as a common framework for both topological spaces and the underlying algebraic structures of their topologies – locales.
- The category of locales (resp. topological spaces) appeared to be isomorphic to a full (resp. co)reflective subcategory of the category of topological systems, which gave rise to the so-called system *localification* (resp. *spatialization*) procedure.

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Affine systems

- In 1996, Y. Diers has come out with the concept of *affine set*, which included topological spaces as a particular example.
- The respective notion of *affine system* extends topological systems of S. Vickers, and also state property systems of D. Aerts.
- The category of affine sets is isomorphic to a full coreflective subcategory of the category of affine systems, giving an affine analogue of the spatialization procedure for topological systems.

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Localification procedure for affine systems

- This talk shows the necessary and sufficient condition for the dual category of the variety of algebras, whose objects underly the structure of affine sets, to be isomorphic to a full reflective subcategory of the category of affine systems, giving an affine analogue of the localification procedure for topological systems.
- One obtains a restatement of the *sobriety-spatiality equivalence* for affine sets, which is patterned after the equivalence between the categories of sober topological spaces and spatial locales.
- The existence of the localification procedure for affine systems induces, moreover, their category to be essentially algebraic.

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Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly, proper or empty) class Λ.

- An *Ω-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (*n_λ-ary primitive operations* on A).
- An *Ω-homomorphism* $(A_1, (\omega_\lambda^{A_1})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda \in \Lambda})$ is a map $A_1 \xrightarrow{\varphi} A_2$ such that $\varphi \circ \omega_\lambda^{A_1} = \omega_\lambda^{A_2} \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Varieties and algebras

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called *algebras* (resp. *homomorphisms*).

Examples of varieties

Example 3

1. **CSLat**$(\bigvee)$ is the variety of $\bigvee$*-semilattices*, and **CSLat**$(\bigwedge)$ is the variety of $\bigwedge$*-semilattices*.
2. **Frm** is the variety of *frames*.
3. **CBAlg** is the variety of *complete Boolean algebras*.
4. **CSL** is the variety of *closure semilattices*, i.e., $\bigwedge$-semilattices, with the singled out bottom element.

Affine spaces

Definition 4

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, where $\mathbf{B}$ is a variety of algebras, $\mathbf{AfSpc}(T)$ is the concrete category over $\mathbf{X}$, whose

objects (*T-affine spaces* or *T-spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX;

morphisms (*T-affine morphisms* or *T-morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ with the property that $(Tf)^{op}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.

Examples

Example 5

Given a variety $\mathbf{B}$, every subcategory $\mathbf{S}$ of $\mathbf{B}^{op}$ induces a functor $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{B}^{op}$, $\mathcal{P}_{\mathbf{S}}((X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)) = B_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} B_2^{X_2}$, where $(\mathcal{P}_{\mathbf{S}}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

Example 6

1. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSpc}(\mathcal{P}_2) = \mathbf{Top}$ (topological spaces).
2. If $\mathbf{B} = \mathbf{CSL}$, then $\mathbf{AfSpc}(\mathcal{P}_2) = \mathbf{Cls}$ (closure spaces).
3. $\mathbf{AfSpc}(\mathcal{P}_B)$ is the category $\mathbf{Af\,Set}(B)$ of affine sets of Y. Diers.
4. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSpc}(\mathcal{P}_{\mathbf{S}}) = \mathbf{S}\text{-}\mathbf{Top}$ (variable-basis lattice-valued topological spaces of S. E. Rodabaugh).

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Affine systems

Definition 7

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, $\mathbf{AfSys}(T)$ is the comma category $(T \downarrow 1_{\mathbf{B}^{op}})$, concrete over the product category $\mathbf{X} \times \mathbf{B}^{op}$, whose

objects (T*-affine systems* or T*-systems*) are triples (X, κ, B), made by $\mathbf{B}^{op}$-morphisms $TX \xrightarrow{\kappa} B$;

morphisms (T*-affine morphisms* or T*-morphisms*) $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)$ are $\mathbf{X} \times \mathbf{B}^{op}$-morphisms $(X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)$, making the next diagram commute

$$\begin{tikzcd} TX_1 \arrow[r, "Tf"] \arrow[d, "\kappa_1"'] & TX_2 \arrow[d, "\kappa_2"] \\ B_1 \arrow[r, "\varphi"'] & B_2. \end{tikzcd}$$

Examples

Example 8

1. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSys}(\mathcal{P}_2) = \mathbf{TopSys}$ (topological systems of S. Vickers).
2. If $\mathbf{B} = \mathbf{Set}$, then $\mathbf{AfSys}(\mathcal{P}_B) = \mathbf{Chu}_B$ (Chu spaces over a set B of P.-H. Chu).

Definition 9

A T-system (X, κ, B) is called *separated* provided that $TX \xrightarrow{\kappa} B$ is an epimorphism in $\mathbf{B}^{op}$. $\mathbf{AfSys}_s(T)$ is the full subcategory of $\mathbf{AfSys}(T)$ of separated T-systems.

Example 10

For $\mathbf{B} = \mathbf{CSL}$, $\mathbf{AfSys}_s(\mathcal{P}_2) = \mathbf{SP}$ (state property systems of D. Aerts).

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Affine spatialization procedure

Theorem 11

1. $\mathbf{AfSpc}(T) \overset{E}{\hookrightarrow} \mathbf{AfSys}(T)$, $E((X_1,\tau_1) \xrightarrow{f} (X_2,\tau_2)) = (X_1, e_{\tau_1}^{op}, \tau_1) \xrightarrow{(f,\varphi)} (X_2, e_{\tau_2}^{op}, \tau_2)$ *is a full embedding, with* e_{τ_i} *the inclusion* $\tau_i \hookrightarrow TX_i$, *and* φ^{op} *the restriction* $\tau_2 \xrightarrow{(Tf)^{op}|_{\tau_2}^{\tau_1}} \tau_1$.
2. E *has a right-adjoint-left-inverse* $\mathbf{AfSys}(T) \xrightarrow{Spat} \mathbf{AfSpc}(T)$, $Spat((X_1,\kappa_1,B_1) \xrightarrow{(f,\varphi)} (X_2,\kappa_2,B_2)) = (X_1, \kappa_1^{op}(B_1)) \xrightarrow{f} (X_2, \kappa_2^{op}(B_2))$.
3. $\mathbf{AfSpc}(T)$ *is isomorphic to a full (regular mono)-coreflective subcategory of* $\mathbf{AfSys}(T)$.

Consequences

Theorem 12

E and Spat restrict to $\mathbf{AfSpc}(T) \xhookrightarrow{\overline{E}} \mathbf{AfSys}_s(T)$ *and* $\mathbf{AfSys}_s(T) \xrightarrow{\overline{Spat}} \mathbf{AfSpc}(T)$*, providing an equivalence between the categories* $\mathbf{AfSpc}(T)$ *and* $\mathbf{AfSys}_s(T)$ *such that* $\overline{Spat}\,\overline{E} = 1_{\mathbf{AfSpc}(T)}$.

Corollary 13

$\mathbf{AfSpc}(T)$ *is the amnestic modification of* $\mathbf{AfSys}_s(T)$.

Example 14

1. **Top** is isomorphic to a full (regular mono)-coreflective subcategory of **TopSys** (system spatialization procedure of S. Vickers).
2. The categories **Cls** and **SP** are equivalent.

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Categories of affine spaces and systems

Theorem 15

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, *the concrete category* $(\mathbf{AfSpc}(T), |-|)$ *is topological over the ground category* $\mathbf{X}$.

Theorem 16

Suppose $\mathbf{X}$ *is (Epi, Mono-Source)-factorizable, and* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ *preserves epimorphisms. Then the concrete category* $(\mathbf{AfSys}(T), |-|)$ *is essentially algebraic over the ground category* $\mathbf{X} \times \mathbf{B}^{op}$.

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Category of affine systems

Theorem 17

Suppose that $\mathbf{X}$ *is (Epi, Mono-Source)-factorizable,* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ *preserves epimorphisms, and, moreover, the following three equivalent conditions hold:*

1. $\mathbf{B}$ *has the (Epi, Mono)-diagonalization property;*
2. $\mathit{ExtrEpi}(\mathbf{B}) = \mathit{Epi}(\mathbf{B})$*;*
3. *epimorphisms in* $\mathbf{B}$ *are surjective.*

Then the concrete category $(\mathbf{AfSys}(T), |-|)$ *is algebraic over the ground category* $\mathbf{X} \times \mathbf{B}^{op}$.

Affine localification procedure

Proposition 18

$\mathbf{AfSys}(T) \xrightarrow{Loc} \mathbf{B}^{op}$, $Loc((X_1,\kappa_1,B_1) \xrightarrow{(f,\varphi)} (X_2,\kappa_2,B_2)) = B_1 \xrightarrow{\varphi} B_2$ is a functor.

Theorem 19

Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, the following are equivalent.

1. *There exists an adjoint situation $(\eta,\varepsilon) : T \dashv Pt : \mathbf{B}^{op} \to \mathbf{X}$.*
2. *There exists a full embedding $\mathbf{B}^{op} \overset{E}{\hookrightarrow} \mathbf{AfSys}(T)$ such that Loc is a left-adjoint-left-inverse to E. $\mathbf{B}^{op}$ is then isomorphic to a full reflective subcategory of $\mathbf{AfSys}(T)$.*

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Proof of Theorem 19

(1) ⇒ (2)

- The required embedding functor $\mathbf{B}^{op} \overset{E}{\hookrightarrow} T$ can be defined by $E(B_1 \xrightarrow{\varphi} B_2) = (PtB_1, \varepsilon_{B_1}, B_1) \xrightarrow{(Pt\varphi, \varphi)} (PtB_2, \varepsilon_{B_2}, B_2)$.
- Given a T-system (X, κ, B), straightforward calculations show that $(X, \kappa, B) \xrightarrow{(f := Pt\kappa \circ \eta_X, 1_B)} ((PtB, \varepsilon_B, B) = ELoc(X, \kappa, B))$ provides an E-universal arrow for (X, κ, B).

(2) ⇒ (1)

Given an adjunction $Loc \dashv E : \mathbf{B}^{op} \to \mathbf{AfSys}(T)$, $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ is the composition of the left adjoint functors $\mathbf{X} \to \mathbf{AfSpc}(T)$ (*indiscrete functor*), $\mathbf{AfSpc}(T) \overset{E}{\hookrightarrow} \mathbf{AfSys}(T)$, and $\mathbf{AfSys}(T) \xrightarrow{Loc} \mathbf{B}^{op}$.

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Examples

Remark 20

Every functor $\mathbf{Set} \xrightarrow{\mathcal{P}_B} \mathbf{B}^{op}$ *has a right adjoint* $\mathbf{B}^{op} \xrightarrow{Pt_B} \mathbf{Set}$, $Pt_B(B_1 \xrightarrow{\varphi} B_2) = \mathbf{B}(B_1, B) \xrightarrow{Pt_B\varphi} \mathbf{B}(B_2, B)$, $(Pt_B\varphi)(p) = p \circ \varphi^{op}$.

Example 21

- **Loc** is isomorphic to a full reflective subcategory of **TopSys**, which gives the system localification procedure of S. Vickers.
- $\mathbf{B}^{op}$ is isomorphic to a full reflective subcategory of $\mathbf{AfSys}(\mathcal{P}_B)$.

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Counterexample

Proposition 22

Take a functor $\mathbf{Set} \times \mathbf{B}^{op} \xrightarrow{T := \mathcal{P}_{\mathbf{B}^{op}}} \mathbf{B}^{op}$. *Suppose that there is a* $\mathbf{B}$-*algebra* B*, whose underlying set is finite with at least two elements, e.g., has the cardinality* n, $n \geqslant 2$. *Then* T *has no right adjoint.*

Proof.

- If T has a right adjoint, then T preserves coproducts.
- For a singleton set 1, $T((1, A) \coprod (1, A)) = T((1 \uplus 1, A \times A)) = (A \times A)^{(1 \uplus 1)}$ and $T(1, A) \times T(1, A) = A^1 \times A^1$.
- Since $T((1, A) \coprod (1, A)) \cong T(1, A) \times T(1, A)$, one gets $n^4 = Card((A \times A)^{(1 \uplus 1)}) = Card(A^1 \times A^1) = n^2$, i.e., contradiction.

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Consequence

Proposition 23

Suppose that the category $\mathbf{X}$ *is (Epi, Mono-Source)-factorizable. If there exists a full embedding* $\mathbf{B}^{op} \overset{E}{\hookrightarrow} \mathbf{AfSys}(T)$ *such that* Loc *is a left-adjoint-left-inverse to* E*, then the concrete category* $(\mathbf{AfSys}(T), |-|)$ *is essentially algebraic over* $\mathbf{X} \times \mathbf{B}^{op}$*.*

Algebras versus affine spaces

- Let $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ be a functor, which has a right adjoint.
- The adjoint situations $\mathbf{AfSpc}(T) \underset{Spat}{\overset{E_S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{AfSys}(T) \underset{E_L}{\overset{Loc}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{B}^{op}$

  give rise to the adjoint situation $\mathbf{AfSpc}(T) \underset{PT:=SpatE_L}{\overset{O:=LocE_S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{B}^{op}$,

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Affine sobriety and spatiality

Definition 24

Sob is the full subcategory of **AfSpc**(T), which contains T-spaces (X,τ) such that $(X,\tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} PTO(X,\tau)$ is an isomorphism.

Definition 25

Spat is the full subcategory of $\mathbf{B}^{op}$, which contains **B**-algebras B such that $OPTB \xrightarrow{\hat{\varepsilon}_B} B$ is an isomorphism.

Proposition 26

$\mathbf{AfSpc}(T) \underset{PT}{\overset{O}{\rightleftarrows}} \mathbf{B}^{op}$ *restricts to an equivalence* $\mathbf{Sob} \underset{PT}{\overset{O}{\rightleftarrows}} \mathbf{Spat}$.

Affine sobriety and spatiality

Definition 24

Sob is the full subcategory of $\mathbf{AfSpc}(T)$, which contains T-spaces (X,τ) such that $(X,\tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} PTO(X,\tau)$ is an isomorphism.

Definition 25

Spat is the full subcategory of $\mathbf{B}^{op}$, which contains $\mathbf{B}$-algebras B such that $OPTB \xrightarrow{\hat{\varepsilon}_B} B$ is an isomorphism.

Proposition 26

$\mathbf{AfSpc}(T) \underset{PT}{\overset{O}{\rightleftarrows}} \mathbf{B}^{op}$ *restricts to an equivalence* $\mathbf{Sob} \underset{PT}{\overset{O}{\rightleftarrows}} \mathbf{Spat}$.

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Affine sobriety and spatiality

Example 27

- There exists the adjoint situation $O \dashv PT : \mathbf{Loc} \to \mathbf{Top}$ and its respective equivalence between the categories **Spat** (*spatial locales*) and **Sob** (*sober topological spaces*).
- There exists the adjoint situation $O \dashv PT : \mathbf{B}^{op} \to \mathbf{AfSet}(A)$ and its respective equivalence $\mathbf{Spat} \sim \mathbf{Sob}$ (Y. Diers).

Separated affine spaces

Definition 28

A T-space (X, τ) is said to be *separated* provided that $(X, \tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} PTO(X, \tau)$ is a monomorphism. $\mathbf{AfSps}_s(T)$ is the full subcategory of $\mathbf{AfSpc}(T)$ of separated T-spaces.

Theorem 29

Let $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ be a functor. If $\mathbf{X}$ has a proper $(\mathcal{E}, Mono)$-factorization system, where Mono is the class of $\mathbf{X}$-monomorphisms, then $\mathbf{AfSps}_s(T)$ is an epireflective subcategory of $\mathbf{AfSpc}(T)$.

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Examples

Example 30

If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{AfSps}_s(\mathcal{P}_2) = \mathbf{Top}_0$ (T_0 topological spaces).

Example 31

Since the category **Set** has a proper (Epi, Mono)-factorization system, the above theorem is applicable to every functor $\mathbf{Set} \xrightarrow{\mathcal{P}_B} \mathbf{B}^{op}$.

- $\mathbf{Top}_0$ is a reflective subcategory of **Top**.
- $\mathbf{Cls}_0$ is a reflective subcategory of **Cls**.
- $\mathbf{AfSet}_s(A)$ is a reflective subcategory of $\mathbf{AfSet}(A)$ (Y. Diers).

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Spatial affine systems

Definition 32

A T-system (X, κ, B) is called *spatial* provided that there exists a T-space (X, τ) such that (X, κ, B) is isomorphic to $E_S(X, \tau)$.

Proposition 33

Given a T-system (X, κ, B), the following are equivalent:

1. *(X, κ, B) is spatial;*
2. *the T-morphism $(E_S Spat(X, \kappa, B) = (X, e^{op}_{\kappa^{op}(B)}, B)) \xrightarrow{(1_X, \kappa)} (X, \kappa, B)$ is an isomorphism;*
3. *the **B**-homomorphism $B \xrightarrow{\kappa^{op}} \kappa^{op}(B)$ is an isomorphism;*
4. *the **B**-homomorphism $B \xrightarrow{\kappa^{op}} TX$ is injective.*

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Localic affine systems

Definition 34

A T-system (X, κ, B) is called *localic* provided that there exists a $\mathbf{B}$-algebra B such that (X, κ, B) is isomorphic to $E_L B$.

Proposition 35

Given a T-system (X, κ, B), the following are equivalent:

1. *(X, κ, B) is localic;*
2. *the T-morphism $(X, \kappa, B) \xrightarrow{(Pt\kappa \circ \eta_X, 1_B)} (E_L Loc(X, \kappa, B) = (PtB, \varepsilon_B, B))$ is an isomorphism;*
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Spatial and localic affine systems

Definition 36

SpaLoc is the full subcategory of **AfSys**(T) of T-systems, which are spatial and localic.

Theorem 37

$$\begin{tikzcd} \mathbf{Spat} \arrow[r, hook, bend right, "\overline{E_L}"'] & \mathbf{SpaLoc} \arrow[l, "\overline{Loc}"'] \arrow[r, "\overline{Spat}"'] & \mathbf{Sob} \arrow[l, hook', bend right, "\overline{E_S}"'] \end{tikzcd}$$ *are equivalences.*

The above theorem provides an internalization of the sobriety-spatiality equivalence into the category of affine systems.

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Spaces versus systems

- The category **TopSys** of topological systems of S. Vickers embeds the category **Loc** of locales (resp. **Top** of topological spaces) as a full (resp. co)reflective subcategory.
- The category **AfSys**(T) of affine systems (motivated by affine sets of Y. Diers) embeds the category $\mathbf{B}^{op}$ of the underlying algebras of affine structures (resp. **AfSpc**(T) of affine spaces) as a full (resp. co)reflective subcategory.

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Sobriety and spatiality

- While the embedding of **AfSpc**(T) into **AfSys**(T) is always possible, the embedding of $\mathbf{B}^{op}$ requires the existence of a right adjoint for the respective functor T.
- The obtained embeddings allowed us to restate the equivalence between the categories of sober topological spaces and spatial locales in the language of algebras and affine spaces, and to internalize this equivalence into the category of affine systems.

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Thank you for your attention!