The geometry of positive commutative totally ordered monoids

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Our topic

Definition

$(L; \leqslant, +, 0)$ is a totally ordered monoid (tomonoid) if:

(T1) $(L; +, 0)$ is monoid,

(T2) $\leqslant$ is a compatible total order:
$a \leqslant b$ and $c \leqslant d$ imply $a + c \leqslant b + d$.

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A tomonoid is called
commutative if so is $+$,
positive if 0 is the bottom element,
finitely generated if so is the monoid $(L; +, 0)$.

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Any commutative monoid generated by at most n elements is isomorphic to a quotient of $\mathbb{N}^n$.

Congruences on $\mathbb{N}^n$

Theorem (Eilenberg, Schützenberger, *J. Alg.* 1969)

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Theorem (Hirshfeld, *Techn. Rep.* 1993)

Every congruence on a finitely generated commutative monoid is semilinear, that is, the union of finitely many sets of the form

$$(v, w) + \operatorname{span}\{(s_1, t_1), \ldots, (s_l, t_l)\}.$$

Let $(L; \leqslant, +, 0)$ be a
finite, positive, commutative (f.p.c.) tomonoid.

Let $\sim$ be a congruence on $\mathbb{N}^n$
such that the quotient $\langle \mathbb{N}^n \rangle_\sim$ is isomorphic to $(L; +, 0)$.

Congruences on $\mathbb{N}^n$: the case of tomonoids

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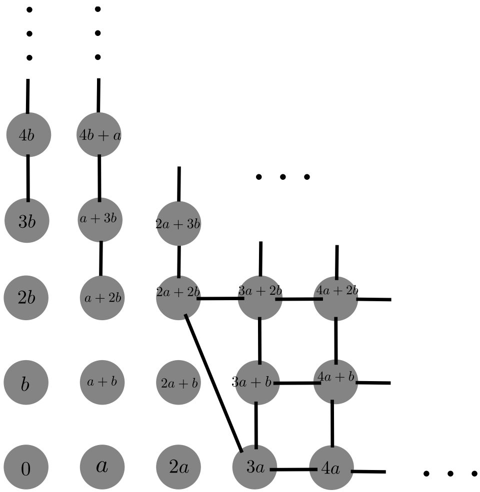
Question

Can we be more specific about $\sim$ in this case?

Example

Monoid reduct of a 10-element f.p.c. tomonoid



Monomial preorders and f.p.c. tomonoids

Let $(L; \leqslant, +, 0)$ be a f.p.c. tomonoid.
Let $\varphi \colon \mathbb{N}^n \to L$ be a surjective monoid homomorphism.

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Observation

$\preccurlyeq$ is a preorder on $\mathbb{N}^n$, which is ...

- compatible: $a \preccurlyeq b$ implies $a + c \preccurlyeq b + c$;
- positive: $0 \prec a$ for all a;
- total: $a \preccurlyeq b$ or $b \preccurlyeq a$ for each a, b.

We call such preorders monomial.

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Observation

There is a mutual correspondence between monomial preorders and finitely generated, positive, commutative tomonoids.

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Lemma

Let $\approx$ be the symmetrisation of $\preccurlyeq$:

$$a \approx b \quad \text{if } a \preccurlyeq b \text{ and } b \preccurlyeq a.$$

Then $\approx$ is the congruence on $\mathbb{N}^n$ leading to $(L; +, 0)$.

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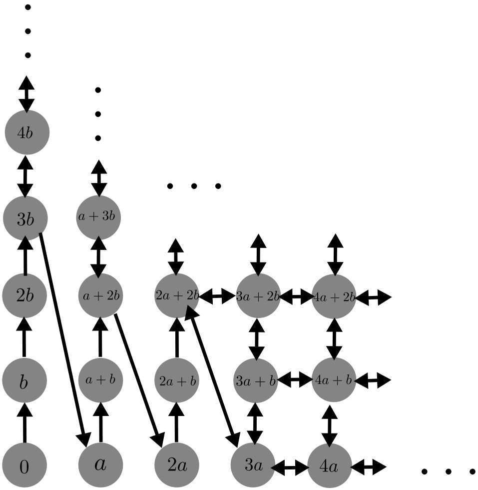
... and the total order:

Lemma

$\langle a \rangle_{\preccurlyeq} \leqslant \langle b \rangle_{\preccurlyeq}$ in L if and only if $a \preccurlyeq b$.

Example

10-element
f.p.c. tomonoid



Congruences on $\mathbb{N}^n$ associated with tomonoids

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Call a downward closed subset of $\mathbb{N}^n$ a $\trianglelefteq$-ideal.

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Proposition

Let $\sim$ be a finite congruence on $\mathbb{N}^n$ such that the quotient $\langle \mathbb{N}^n \rangle_\sim$ is the monoidal reduct of an f.p.c. tomonoid.

- Each finite $\sim$-class consists of pairwise incomparable elements.

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- Each finite $\sim$-class consists of pairwise incomparable elements.
  The union of all finite $\sim$-classes is a finite $\trianglelefteq$-ideal.
- Each infinite $\sim$-class is the union of finitely many sets of the form
  $$a + \{u_1, \ldots, u_k\}^\star,$$
  where $u_1, \ldots, u_k$ are unit vectors of $\mathbb{N}^n$.

The Archimedean generator classes

Let $\preccurlyeq$ be a monomial preorder.

For $a, b \in \mathbb{N}^n$, we define

$$a \prec\!\!\prec b \quad \text{if } k\,a \prec b \text{ for all } k \geqslant 1.$$

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For $a, b \in \mathbb{N}^n$, we define

$$a \lll b \quad \text{if } k\,a \prec b \text{ for all } k \geqslant 1.$$

Let $\mathcal{U}(\mathbb{N}^n) = \{u_1, \ldots, u_n\}$ be the unit vectors.

Let $A_{\preccurlyeq} = (U_1, \ldots, U_m)$ be the ordered partition of $\mathcal{U}(\mathbb{N}^n)$ into Archimedean classes. This means

$$u \lll v \quad \text{iff } u \in U_i \text{ and } v \in U_j \text{ such that } i < j.$$

The support

For $a \in \mathbb{N}^n \setminus \{\bar{0}\}$, we let $s(a)$ be the smallest i such that $u \trianglelefteq a$ for some $u \in U_i$.

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Definition

The support of $\preccurlyeq$ is

$$\begin{aligned} S_{\preccurlyeq} &= \{a \in \mathbb{N}^n \colon \; a = \bar{0}, \\ &\quad \text{or } a - u \prec a \text{ for some } u \trianglelefteq a \text{ such that } u \in U_{s(a)}\} \end{aligned}$$

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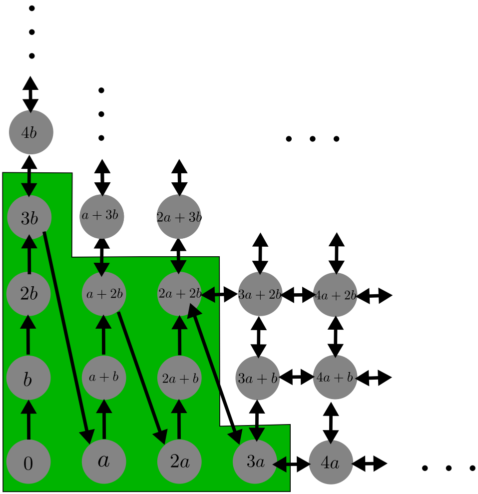
Proposition

$S_{\preccurlyeq}$ is a finite $\trianglelefteq$-ideal of $\mathbb{N}^n$.

Example

Support in green;

Archimedean generator classes: $(\{b\}, \{a\})$



The support and the finite classes

We define

$$\mathring{S}_{\preccurlyeq} = \{a \in S_{\preccurlyeq} : \ a = \bar{0}, \\ \text{or } a + u \in S_{\preccurlyeq} \text{ for all } u \in U_j \text{ such that } j \leqslant s(a)\}.$$

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$S_{\preccurlyeq}$ includes all finite classes.

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Theorem

$S_{\preccurlyeq}$ includes all finite classes.

In fact, $\mathring{S}_{\preccurlyeq}$ is the union of the finite classes.

Let $a \in S_{\preccurlyeq} \setminus \mathring{S}_{\preccurlyeq}$. We call

$$\sigma_{S_{\preccurlyeq}}(a) \; = \; a + (U_1 \cup \ldots \cup U_j)^{\star}$$

the *segment* of a, where $j \in \{1, \ldots, s(a)\}$ is largest such that $a + u \notin S_{\preccurlyeq}$ for some $u \in U_j$.

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$S_{\preccurlyeq}$ has a non-empty intersection with each infinite class.

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Theorem

$S_{\preccurlyeq}$ has a non-empty intersection with each infinite class.

In fact, for each infinite class B we have

$$B \;=\; \bigcup_{a \in B \cap S_{\preccurlyeq}} \sigma_{S_{\preccurlyeq}}(a).$$

With a finite monomial preorder on $\mathbb{N}^n$, we may associate

(1) its Archimedean generator classes $A_{\preccurlyeq} = (U_1, \ldots, U_m)$;

(2) its support $S_{\preccurlyeq} \subseteq \mathbb{N}^n$.

The geometry of tomonoid-induced congruences on $\mathbb{N}^n$

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(1) its Archimedean generator classes $A_{\preccurlyeq} = (U_1, \ldots, U_m)$;

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Let $\approx$ be the associated monoid congruence on $\mathbb{N}^n$. Then

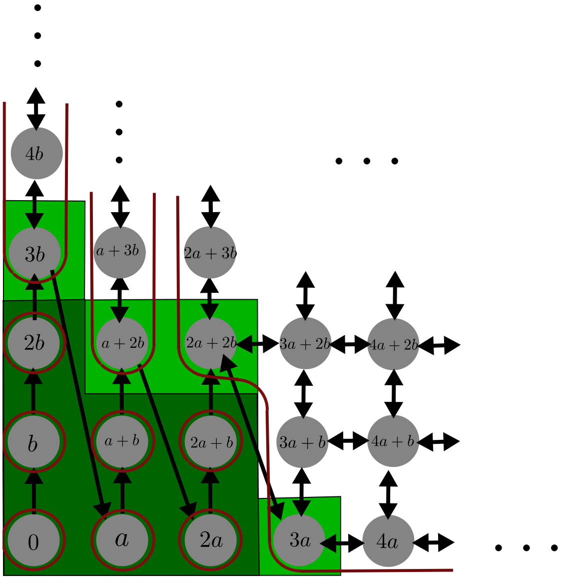
- each finite class is a subset of $\mathring{S}_{\preccurlyeq}$ consisting of pairwise incomparable elements;
- each infinite class is a finite union of sets of the form $\sigma_{S_{\preccurlyeq}}(a),\ a \in S_{\preccurlyeq} \setminus \mathring{S}_{\preccurlyeq}$.

Example

$S_{\preccurlyeq}$
in green;

$\mathring{S}_{\preccurlyeq}$
in dark green;

each class
framed



Specifying the total order

Corollary

Any finite monomial preorder $\preccurlyeq$ is uniquely determined by its restriction to its support $S_{\preccurlyeq}$.

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Question

How can we describe $\preccurlyeq$ on $S_{\preccurlyeq}$?

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How can we describe $\preccurlyeq$ on $S_{\preccurlyeq}$?

Let $\mathcal{D}(S) = \{b - a \in \mathbb{Z}^n : a, b \in S\}$.

Definition

We define

$$F_{\preccurlyeq} \;=\; \{z \in \mathcal{D}(S_{\preccurlyeq}) : a \preccurlyeq b \text{ whenever } b - a = z\}.$$

The triple $(A_{\preccurlyeq}, S_{\preccurlyeq}, F_{\preccurlyeq})$ is called the *direction f-cone* of $\preccurlyeq$.

Properties of direction f-cones

Theorem

Let $\preccurlyeq$ be a finite monomial preorder. Then the direction f-cone (A, S, F) of $\preccurlyeq$ has the following properties:

(Cf1) For each $z \in \mathcal{D}(S)$, $z \trianglerighteq 0$ implies $z \in F$ and, if $z \neq 0$, $-z \notin F$.

(Cf2) Let $(x_1, \ldots, x_k)$, $k \geqslant 2$, be an addable k-tuple of elements of F whose sum is in $\mathcal{D}(S)$. Then $x_1 + \ldots + x_k \in F$.

(Cf3) For each $z \in \mathcal{D}(S)$, either $z \in F$ or $-z \in F$.

(Cf4) Let $a, b \in S$ be such that $a \lll_A b$. Then $a - b \notin F$.

Here, $(x_1, \ldots, x_k)$ to be addable means that

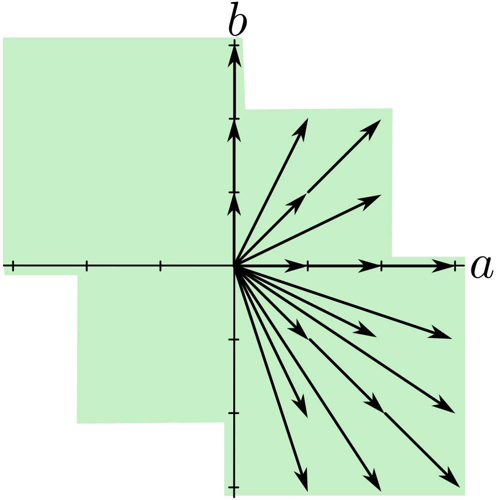
$$(x_1 + \ldots + x_k)^- + x_1 + \ldots + x_i \;\trianglerighteq\; \bar{0}$$

for $i = 0, \ldots, k$.

Example

Direction cone $F_{\preccurlyeq}$;

$\mathcal{D}(S_{\preccurlyeq})$ in light green



The converse way

Theorem

Let $\mathcal{C} = (A, S, F)$, where

A is an ordered partition of $\mathcal{U}(\mathbb{N}^n)$,

S is a finite $\trianglelefteq$-ideal of $\mathbb{N}^n$ including $\mathcal{U}(\mathbb{N}^n)$,

$F \subseteq \mathcal{D}(S)$.

Assume that $\mathcal{C}$ fulfils properties (Cf1)–(Cf4).

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Let $\preccurlyeq_{\mathcal{C}}$ be the smallest preorder such that:

(O1) $a \preccurlyeq_{\mathcal{C}} b$ for any $a, b \in \mathbb{N}^n$ such that $b - a \in F$.

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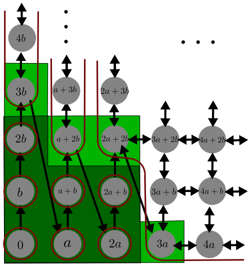
Then $\preccurlyeq_{\mathcal{C}}$ is a finite monomial preorder.

Moreover, any finite monomial preorder is an extension of a monomial preorder arising in this way.

The construction of f.p.c. tomonoids



Starting from

- a partion A of the unit vectors of $\mathbb{N}^n$;
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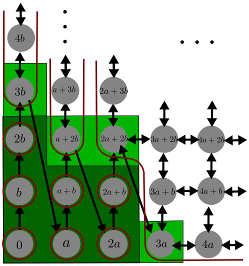

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Any finite, positive, commutative tomonoid is a quotient of a tomonoid arising in this way.