Independence of algebras

Erhard Aichinger and Peter Mayr

Department of Algebra
Johannes Kepler University Linz, Austria

June 2015, AAA90

Supported by the Austrian Science Fund (FWF)
P24077 and P24285

Outline

We will study:

- relation between $\mathrm{Clo}_k(\mathbf{A}), \mathrm{Clo}_k(\mathbf{B})$ and $\mathrm{Clo}_k(\mathbf{A} \times \mathbf{B})$.
- relation between $\mathbf{F}_{V(\mathbf{A})}(k) \times \mathbf{F}_{V(\mathbf{B})}(k)$ and $\mathbf{F}_{V(\mathbf{A} \times \mathbf{B})}(k)$.
- relation between $V(\mathbf{A}), V(\mathbf{B})$ and $V(\mathbf{A}) \vee V(\mathbf{B})$.

Term functions on direct products

Question

How do the term functions of $\mathbf{A} \times \mathbf{B}$ depend on the term functions of $\mathbf{A}$ and $\mathbf{B}$?

Proposition

Let $\mathbf{A}$, $\mathbf{B}$ be similar algebras, $k \in \mathbb{N}$, and define

$$\begin{array}{rrcl} \phi : & \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) & \longrightarrow & \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B}) \\ & t^{\mathbf{A} \times \mathbf{B}} & \longmapsto & (t^{\mathbf{A}},\ t^{\mathbf{B}}). \end{array}$$

Then ϕ is a subdirect embedding.

Proposition

$\mathbf{A}$, $\mathbf{B}$ from a cp variety, $k \in \mathbb{N}$. Then for all k-ary terms s, t:

$$(s^{\mathbf{A}}, t^{\mathbf{B}}) \in \mathrm{Im}(\phi) \Longleftrightarrow V(\mathbf{A}) \cap V(\mathbf{B}) \models s \approx t.$$

Disjoint varieties

$$\begin{array}{rcrcl} \phi & : & \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) & \longrightarrow & \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B}) \\ & & t^{\mathbf{A}\times\mathbf{B}} & \longmapsto & (t^{\mathbf{A}},\ t^{\mathbf{B}}). \end{array}$$

If **A**, **B** are from a cp variety, then

$$\begin{array}{rcl} (s^{\mathbf{A}}, t^{\mathbf{B}}) \in \mathrm{Im}(\phi) & \Leftrightarrow & \exists u : u^{\mathbf{A}} = s^{\mathbf{A}} \text{ and } u^{\mathbf{B}} = t^{\mathbf{B}} \\ & \Leftrightarrow & V(\mathbf{A}) \cap V(\mathbf{B}) \models s \approx t. \end{array}$$

Definition

V_1 and V_2 are *disjoint* if $V_1 \cap V_2 \models x \approx y$.

Corollary

A, **B** from a cp variety, $k \geq 2$. Then ϕ is an isomorphism from $\mathrm{Clo}_k(\mathbf{A} \times \mathbf{B})$ to $\mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B}) \iff V(\mathbf{A})$ and $V(\mathbf{B})$ are disjoint.

History (1955 – 1969)

Definition [Foster, 1955]

A sequence $(V_1, \ldots, V_n)$ of subvarities of W is *independent* if there is a term $t(x_1, \ldots, x_n)$ such that $\forall i \in [n] : V_i \models t(x_1, \ldots, x_n) \approx x_i$.

Example [Grätzer, Lakser, Płonka, 1969]

$$
\begin{array}{lcll}
V_0 & := & \{ & (G,\ f_0(x,y) = x \cdot y,\ f_1(x,y) = x \cdot y^{-1}) \mid \\
 & & & (G, \cdot, {}^{-1}, 1) \text{ is a group}\} \\
V_1 & := & \{ & (L,\ f_0(x,y) = x \vee y,\ f_1(x,y) = x \wedge y) \mid \\
 & & & (L, \vee, \wedge) \text{ is a lattice}\}, \\
t(x,y) & := & & f_1(f_0(x,y), y).
\end{array}
$$

Then

- $V_0 \models f_1(f_0(x,y),y) = (x \cdot y) \cdot y^{-1} \approx x$ and
- $V_1 \models f_1(f_0(x,y),y) = (x \vee y) \wedge y \approx y$.

History (1969)

Theorem [Grätzer, Lakser, Płonka, 1969]

Let V_0 and V_1 be independent subvarieties of W. Then every $\mathbf{A} \in V_0 \vee V_1$ is isomorphic to a direct product $\mathbf{A}_0 \times \mathbf{A}_1$ with $\mathbf{A}_0 \in V_0$ and $\mathbf{A}_1 \in V_1$.

Consequence

Let V_0 and V_1 be independent. Then $(V_0 \vee V_1)_{SI} = (V_0)_{SI} \cup (V_1)_{SI}$.

History (1971)

Theorem [Hu and Kelenson, 1971]

Let $(V_1, \dots, V_n)$ be a sequence of subvarieties of a cp variety W. If for all $i \neq j$, $V_i \cap V_j \models x \approx y$ (V_i and V_j are disjoint), then $(V_1, \dots, V_n)$ is independent.

Proof for $n = 2$:

- Goal: construct $t(x_1, x_2)$ with $V_1 \models t(x_1, x_2) \approx x_1$ and $V_2 \models t(x_1, x_2) \approx x_2$.
- $\phi : \mathbf{F}_{V_1 \vee V_2}(x, y) \to \mathbf{F}_{V_1}(x, y) \times \mathbf{F}_{V_2}(x, y)$, $t/{\sim_{V_1 \vee V_2}} \mapsto (t/{\sim_{V_1}}, t/{\sim_{V_2}})$.
- $\mathrm{Im}(\phi) \leq_{sd} \mathbf{F}_{V_1}(x, y) \times \mathbf{F}_{V_2}(x, y)$.
- Fleischer's Lemma yields $\mathbf{D}$, $\alpha_1 : \mathbf{F}_{V_1}(x, y) \twoheadrightarrow \mathbf{D}$, $\alpha_2 : \mathbf{F}_{V_2}(x, y) \twoheadrightarrow \mathbf{D}$ with

$$\mathrm{Im}(\phi) = \{(f, g) \,|\, \alpha_1(f) = \alpha_2(g)\}.$$

- $|\mathbf{D}| = 1$, hence ϕ is surjective.
- Thus $(x/{\sim_{V_1}}, y/{\sim_{V_2}}) \in \mathrm{Im}(\phi)$, which yields t.

History (2004 – 2013)

Theorem [Jónsson and Tsinakis, 2004]

The join of two independent finitely based varieties is finitely based.

Theorem [Kowalski, Paoli, Ledda, 2013]

Let V_1, V_2 be disjoint subvarieties of W. Then V_1 and V_2 are independent iff $\exists q(x, y, z) : V_1 \models q(x, x, y) \approx y$ and $V_2 \models q(x, y, y) \approx x$.

Product subalgebras

Definition

$\mathbf{C} \leq \mathbf{E} \times \mathbf{F}$ is a *product subalgebra* if $\mathbf{C} = \pi_{\mathbf{E}}(\mathbf{C}) \times \pi_{\mathbf{F}}(\mathbf{C})$.

Proposition

$\mathbf{C} \leq \mathbf{E} \times \mathbf{F}$ is a product subalgebra iff for all a, b, c, d:
$(a, b) \in C$ and $(c, d) \in C \implies (a, d) \in C$.

Definition

$\alpha \in \mathrm{Con}(\mathbf{E} \times \mathbf{F})$ is a *product congruence* if $\alpha = \beta \times \gamma$ for some $\beta \in \mathrm{Con}(\mathbf{E})$ and $\gamma \in \mathrm{Con}(\mathbf{F})$.

Product subalgebras of powers

Theorem [EA and Mayr, 2015]

Let $\mathbf{A}, \mathbf{B}$ be algebras in a congruence permutable variety. We assume that

1. all subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras, and
2. for all $\mathbf{E} \leq \mathbf{A}$ and $\mathbf{F} \leq \mathbf{B}$, all congruences of $\mathbf{E} \times \mathbf{F}$ are product congruences.

Then for all $m, n \in \mathbb{N}_0$, all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras.

Product subalgebras of powers

Theorem [EA and Mayr, 2015]

Let $k \geq 2$, let $\mathbf{A}, \mathbf{B}$ be algebras in a variety with k-edge term. We assume that

1. for all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra, and
2. for all $\mathbf{E} \leq \mathbf{A}$ and $\mathbf{F} \leq \mathbf{B}$, every tolerance of $\mathbf{E} \times \mathbf{F}$ is a product tolerance.

Then for all $m, n \in \mathbb{N}_0$, every subalgebra of $\mathbf{A}^m \times \mathbf{B}^n$ is a product subalgebra.

Direct products and independence

Definition

$\mathbf{A}, \mathbf{B} \in W$ are *independent* $:\Longleftrightarrow$ $V(\mathbf{A})$ and $V(\mathbf{B})$ are independent.

Independence in cp varieties

Proposition (known before 2000)

Let $\mathbf{A}$ and $\mathbf{B}$ be similar algebras. TFAE:

1. $\mathbf{A}$ and $\mathbf{B}$ are independent.
2. For all sets I, J with $|I| \leq |A|^2$ and $|J| \leq |B|^2$, all subalgebras of $\mathbf{A}^I \times \mathbf{B}^J$ are product subalgebras.

If $\mathbf{A}$ and $\mathbf{B}$ lie in a cp variety, then these two items are furthermore equivalent to

3. $V(\mathbf{A})$ and $V(\mathbf{B})$ are disjoint.

Theorem (EA, Mayr, 2015)

Let $\mathbf{A}$, $\mathbf{B}$ be finite algebras in a cp variety. TFAE:

1. $\mathbf{A}$ and $\mathbf{B}$ are independent.
2. All subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras, and all congruences of all subalgebras of $\mathbf{A} \times \mathbf{B}$ are product congruences.
3. All subalgebras of $\mathbf{A}^2 \times \mathbf{B}^2$ are product subalgebras.
4. $HS(\mathbf{A}^2) \cap HS(\mathbf{B}^2)$ contains only one element algebras.

Independence for algebras with edge term

Theorem [EA and Mayr, 2015]

Let $k \geq 2$, and let $\mathbf{A}$, $\mathbf{B}$ be finite algebras in a variety with k-edge term. Then the following are equivalent:

1. $\mathbf{A}$ and $\mathbf{B}$ are independent.
2. For all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra, and for all $E \leq \mathbf{A}, F \leq \mathbf{B}$, every tolerance of $\mathbf{E} \times \mathbf{F}$ is a product tolerance.
3. For all $r, s \in \mathbb{N}$ with $r + s \leq \max(4, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra.

Example - infinite groups

Let p, q be primes, $p \neq q$,
$\mathbf{A} := C_{p^\infty} = \{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{p^n} = 1\}$, $\mathbf{B} := C_{q^\infty}$. Then all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras, but $\mathbf{A}$ and $\mathbf{B}$ are not independent.

Application to polynomial functions

Theorem

Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras in a variety with a 3-edge term, and let $k \in \mathbb{N}$. We assume that every tolerance of $\mathbf{A} \times \mathbf{B}$ is a product tolerance. Let $\psi : \mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B}) \to (A \times B)^{(A \times B)^k}$ be the mapping defined by

$$\psi(f, g)\,((a_1, b_1), \ldots, (a_k, b_k)) := (f(\mathbf{a}), g(\mathbf{b}))$$

for $f \in \mathrm{Pol}_k(\mathbf{A}), g \in \mathrm{Pol}_k(\mathbf{B})$, $\mathbf{a} \in A^k$, and $\mathbf{b} \in B^k$. Then ψ is a bijection from $\mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B})$ to $\mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$.

Application to polynomial functions

Corollary

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras in the variety V, and let $k \in \mathbb{N}$. If either

1. V has a majority term, or
2. V is congruence permutable, and every congruence of $\mathbf{A} \times \mathbf{B}$ is a product congruence,

then for all polynomial functions $f \in \mathrm{Pol}_k(\mathbf{A})$ and $g \in \mathrm{Pol}_k(\mathbf{B})$, there is a polynomial function $h \in \mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$ with $h((a_1, b_1), \ldots, (a_k, b_k)) = (f(\mathbf{a}), g(\mathbf{b}))$ for all $\mathbf{a} \in A^k$ and $\mathbf{b} \in B^k$.

Aichinger, E. and Mayr, P. (2015).
Independence of algebras with edge term.
manuscript, available on arXiv:1504.02663v1[math.RA].

Foster, A. L. (1955).
The identities of—and unique subdirect factorization within—classes of universal algebras.
Math. Z., 62:171–188.

Grätzer, G., Lakser, H., and Płonka, J. (1969).
Joins and direct products of equational classes.
Canad. Math. Bull., 12:741–744.

Hu, T. K. and Kelenson, P. (1971).
Independence and direct factorization of universal algebras.
Math. Nachr., 51:83–99.

Jónsson, B. and Tsinakis, C. (2004).
Products of classes of residuated structures.
Studia Logica, 77(2):267–292.

Kowalski, T., Paoli, F., and Ledda, A. (2013).
On independent varieties and some related notions.
Algebra Universalis, 70(2):107–136.