The asymptotic number of planar, slim, semimodular lattice diagrams

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All lattices and diagrams are assumed to be **finite**!

Slim: $J(L) = \text{Chain}_1 \cup \text{Chain}_2$; $\Rightarrow$ planar. (Grätzer–Knapp 1997).

Semimodular: $a \prec b \Longrightarrow a \vee c \preceq b \vee c$.

Natural example: if $\vec{S}$: $\{1\} = S_0 \subset S_1 \subset \cdots \subset S_n = G$ and $\vec{T}$ are composition series of G), then $(\{S_i \cap T_j\}; \supseteq)$.

Application ($\Rightarrow$ motivation): uniqueness part to the classical Jordan–Hölder theorem:

Theorem (uniqueness for the classical Jordan–Hölder theorem)

$\exists \pi \in S_n$ such that the quotient S_i/S_{i-1} is "down-and-up" perspective to $T_{\pi(i)}/T_{\pi(i)-1}$, for $\forall i$. (Grätzer and Nation, AU 2011). Furthermore, this π is unique. ($\sim$ and Schmidt, AU 2011).

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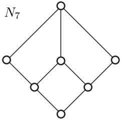
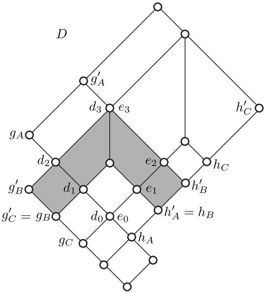
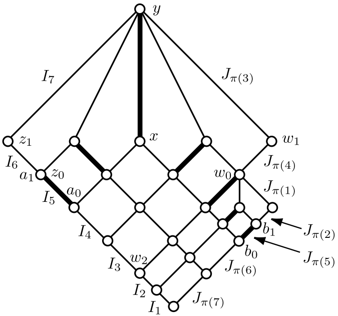
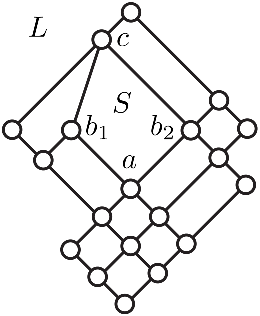


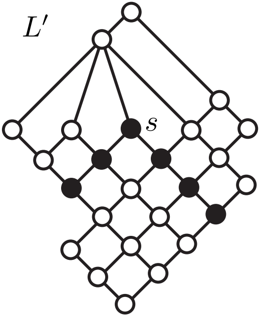




(Unless otherwise suggested, disregard the labeling of the figures.)

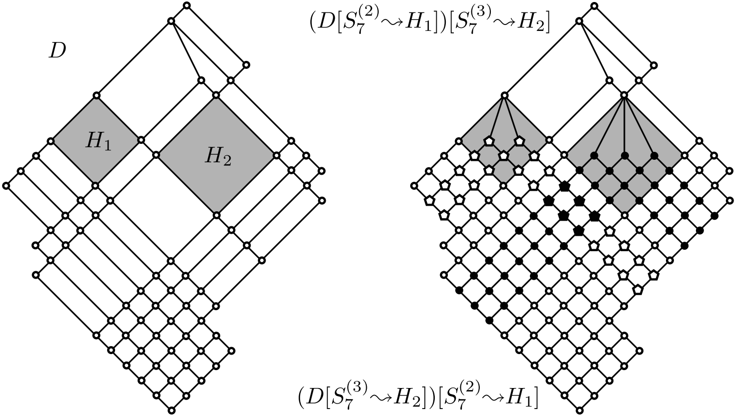
Further examples and a (single) fork extension





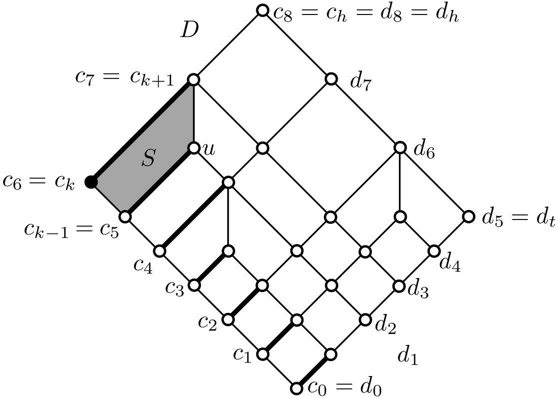
∼ and Schmidt, Order 2012

Still further examples



Multifork extensions at distributive ideals (∼, AU 2014)

A rectangular slim semimodular lattice



This concept is due to Grätzer and Knapp, 2009.

Enumerating slim sm lattices of a given parameter (length, size) n $\approx$ how many ways can the intersections $S_i \cap T_j$ be related, if $\{1\} = S_0 \subset S_1 \subset \cdots \subset S_n = G$ and $\{1\} = T_0 \subset \cdots \subset T_n = G$ are composition series of a finite group.

Enumerating the **diagrams** of these lattices $\approx$ how many ways can the intersection above be related with respect to "$\subseteq$" if the first composition series, $\vec{S}$, is distinguished from the second one.

Definition

We enumerate planar diagrams up to *similarity*. I.e., the "from left to right" ordering of the upper and lower covers of every element is relevant but Eucledian distances ands slopes are not. E.g., N_5 has exactly 2 planar diagrams while M_3 has only 1.

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- M. Erné, J. Heitzig, J. Reinhold: On the number of distributive lattices, Electron. J. Combin. 9/1 (2002), Research Paper 24.
- J. Heitzig, J. Reinhold, Counting finite lattices, Algebra Universalis 48 (2002) 43–53 (of size ≤ 18).
- M.M. Pawar, B.N. Waphare, Enumeration of nonisomorphic lattices with equal number of elements and edges, Indian J. Math. 45 (2003) 315–323

More recent enumerations (slim semimodular lattices) 10'/10

• $\sim$, Ozsvárt, Udvari: How many ways can two composition series intersect, Discr. Math. (2012); recursive formula, lattices computed up to $\ell = 100$, asymptotically $\sim \ell!/2$.

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The number of planar diagrams of slim semimodular lattices of size n is denoted by $N_{\mathrm{d}}(n)$.

Theorem (∼, arxiv:1206.3679, 16 June 2012)

There exists a constant C such that $0 < C < 1$ and $N_{\mathrm{d}}(n)$ is asymptotically $C \cdot 2^n$, that is, $\lim_{n\to\infty}\left(N_{\mathrm{d}}(n)/2^n\right) = C$.

We have also proved: $0.42 \cdot 10^{-57} \le C \le 0.073$.
We conjecture: $0.023 \le C \le 0.073$.

Lattices rather than diagrams of a given size n? The earlier recursion is only effective up to $n \approx 50$. We do not even know if $\lim_{n\to\infty}\left(N_{\mathrm{l}}(n)/N_{\mathrm{l}}(n-1)\right)$ (conjecture: yes), and what is it.

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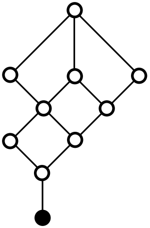
Proof: the initial idea



$\mathrm{rank}_\ell(D) = 0$,
$\mathrm{rank}_r(D) = 0$

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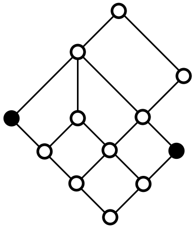
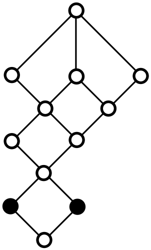
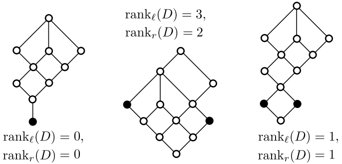
$\mathrm{rank}_r(D) = 2$





$\mathrm{rank}_\ell(D) = 1$,
$\mathrm{rank}_r(D) = 1$

We introduce the left and right rank of a slim sm. diagram. (Slim semimodular) diagrams of size n belong to three categories.

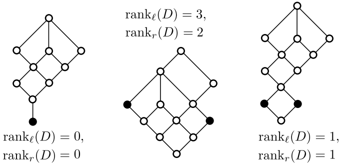


(a) left rank = right rank =0: there are $N_{\mathrm{d}}(n-1)$ of them.

(b) left rank = right rank =1: there are $N_{\mathrm{d}}(n-3)$ of them.

(c) both ranks are > 0: Removing the left corner, $N_{\mathrm{d}}(n-1)$ diagrams remain ??? No, not every diagram of size $n-1$ is obtained in this way. Nevertheless, (a)+(b) and (a)+(c) yield

$$N_{\mathrm{d}}(n-1) + N_{\mathrm{d}}(n-3) \le N_{\mathrm{d}}(n) \le 2 \cdot N_{\mathrm{d}}(n-1).$$



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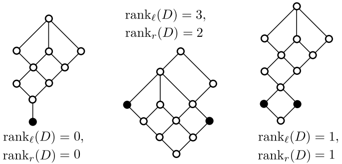

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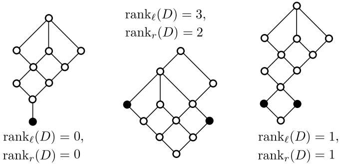

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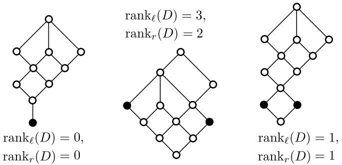
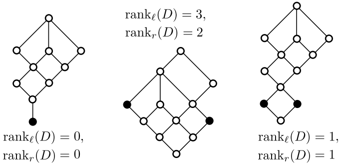


Lemma

D' with $|D'| = n - 1$ is obtained from some D by omitting the left corner iff the left corner of D' is not a coatom.

Proof of the easier direction: length(left corner) decreases by 1. (The other direction is longer.)



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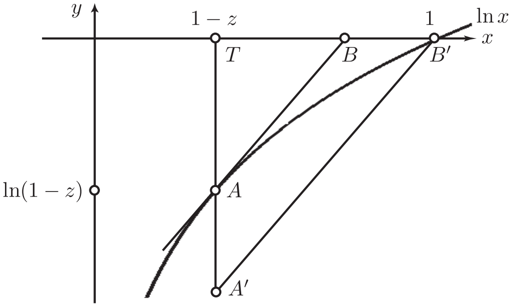
Lemma

If $0 < z \le z_0 < 1$, *then*

$$-\ln(1-z) \le z/(1-z) \le z/(1-z_0).$$



Proof:

With $\kappa_n := N_{\rm d}(n)/N_{\rm d}(n-1)$, $\quad p_n = \prod_{j=m+1}^{n}(\kappa_j/2)$,
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$$0 < s_n = \sum_{j=m+1}^{n} (-\ln(\kappa_j/2)) \leq' \sum_{j=m+1}^{n} (1-\kappa_j/2)/(1-z_0)$$

$$\leq^* \mu \cdot \sum_{j=m+1}^{n} \alpha^{\lceil\sqrt{j-1}\rceil - 2} \leq \mu \cdot \sum_{j=m+1}^{n} \alpha^{\sqrt{j-1}-1} \leq \mu \cdot \sum_{k=m}^{n-1} \alpha^{\sqrt{k}-1}$$

$$= \nu \cdot \sum_{k=m}^{n-1} \alpha^{\sqrt{k}} \leq \nu \cdot \int_{x=m-1}^{n-1} \alpha^{\sqrt{x}} dx \leq \nu \cdot (F(\infty) - F(m-1)),$$

where $F(x) = -2 \cdot \delta^{-2} \cdot (1 + \delta\sqrt{x}) \cdot \alpha^{\sqrt{x}}$. $\exists F(\infty) \Longrightarrow$ Q.e.d.

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