Deciding the existence of a k-wnu operation in polynomial time

Alexandr Kazda

Department of Mathematics
Vanderbilt University
Nashville

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Maltsev conditions

- A strong Maltsev condition is a system of equalities for some operations and variables, eg.

$$p(p(x,y)r(y)) \approx x.$$

- A variety V satisfies the Maltsev condition M if we can replace the operations in M by terms of V and get a system of equalities true in V.
- Linear Maltsev: No nested operations. Example:

$$m(x,x,y) \approx y$$
$$m(y,x,x) \approx y.$$

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The problem

- Given finite **A**, we want to know if $V(\mathbf{A})$ satisfies a given fixed Maltsev condition.
- Example: "Does **A** have a Maltsev term?" (Complexity: Unknown.)
- Known to be EXPTIME-complete for numerous common Maltsev conditions (Freese, Valeriote, Horowitz).
- Restricting **A** to be idempotent often allows us to find a polynomial time algorithm (Freese, Valeriote, Horowitz, K.).
- Big goal: Characterize, the complexity of all versions of the problem for all Maltsev conditions.
- Today's goal: Characterize, the complexity of deciding k-ary weak near unanimity for idempotent algebras when k is fixed.

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Weak near unanimity

- A k-ary weak near unanimity (k-WNU) is any idempotent operation that satisfies

$$t(y, x, \dots, x) \approx t(x, y, \dots, x) \approx \cdots \approx t(x, x, \dots, y).$$

- A k-WNU is an example of a Taylor term.
- Given $\mathbf{A}$ idempotent, we can test in polynomial time whether $\mathbf{A}$ has a Taylor term (Bulatov).

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Local to global

- We say that $\mathbf{A}$ has n-local k-WNUs if for every $\overline{r}, \overline{s} \in A^n$ there exists a term t such that

$$t(s_i, r_i, \dots, r_i) = t(r_i, s_i, r_i, \dots, r_i) = \dots = t(r_i, r_i, \dots, s_i).$$

for all i.

- Translating to relations:

$$\mathrm{Sg}_{\mathbf{A}^n}\left(\begin{pmatrix} \overline{s} \\ \overline{r} \\ \vdots \\ \overline{r} \end{pmatrix}, \begin{pmatrix} \overline{r} \\ \overline{s} \\ \vdots \\ \overline{r} \end{pmatrix}, \dots, \begin{pmatrix} \overline{r} \\ \overline{r} \\ \vdots \\ \overline{s} \end{pmatrix} \right)$$

has a member $(\overline{a}, \overline{a}, \dots, \overline{a})^T$ for some $\overline{a} \in A^n$.

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Local to global, continued

Observation

*If **A** has k-WNU then it has n-local k-WNUs for all n.*

Lemma

*If **A** is idempotent, has a Taylor term and has n-local k-WNUs then **A** also has $(n+1)$-local k-WNUs.*

Corollary

We only need to check for the existence of a Taylor term and for all 1-local k-WNUs. This is polynomial for k fixed.

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Proof for $k = 4$

$$R = \mathrm{Sg}\begin{pmatrix} \overline{s} & \overline{r} & \overline{r} & \overline{r} \\ \overline{r} & \overline{s} & \overline{r} & \overline{r} \\ \overline{r} & \overline{r} & \overline{s} & \overline{r} \\ \overline{r} & \overline{r} & \overline{r} & \overline{s} \\ d & c & c & c \\ c & d & c & c \\ c & c & d & c \\ c & c & c & d \end{pmatrix}.$$

Applying an n-local 4-WNU, we have

$$(\overline{a}, \overline{a}, \overline{a}, \overline{a}, b_1, b_2, b_3, b_4)^T \in R$$

The last four entries can be permuted.

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Loop lemma

Lemma (Barto, Kozik, Niven)

If $\mathbf{G}$ is a Taylor algebra, $E \leq_s \mathbf{G}^2$ and (G, E) has a component of algebraic length 1, then E contains a loop.

In our case, let

$$G = \{(x_1, x_2) \colon \exists y,\ z,\ (\bar{a}, \bar{a}, \bar{a}, \bar{a}, x_1, x_2, y, z)^T \in R\}$$

$$E = \{((x_1, x_2), (x_2, x_3)) \colon \exists z,\ (\bar{a}, \bar{a}, \bar{a}, \bar{a}, x_1, x_2, x_3, z)^T \in R\}.$$

$\overline{G}$ has a Taylor term, E is subdirect, and (G, E) contains a 3-cycle and a 4-cycle in the same component ($\Rightarrow$ algebraic length 1).
The loop lemma gives us e, f such that $(\bar{a}, \bar{a}, \bar{a}, \bar{a}, e, e, e, f) \in R$.

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Final step

$$\begin{pmatrix} \overline{a} \\ \overline{a} \\ \overline{a} \\ \overline{a} \\ e \\ e \\ e \\ f \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{a} \\ \overline{a} \\ \overline{a} \\ e \\ e \\ f \\ e \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{a} \\ \overline{a} \\ \overline{a} \\ e \\ f \\ e \\ e \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{a} \\ \overline{a} \\ \overline{a} \\ f \\ e \\ e \\ e \end{pmatrix} \in R.$$

Apply 4-WNU for e, f and obtain a tuple $(\overline{a}, \overline{a}, \overline{a}, \overline{a}, b, b, b, b)^T$.

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Future outlooks

- The "local to global" idea works for many linear Maltsev conditions in the idempotent case. . .
- . . . but Dmitriy Zhuk has recently shown that "local to global" does not work for the minority operation:

$$t(x, x, y) \approx t(x, y, x) \approx t(y, x, x) \approx y.$$

- Question: How hard is it to decide if $\mathbf{A}$ idempotent has a minority?
- Related problem: Given $\mathbf{A}$ that satisfies a strong Maltsev condition M, produce operations of $\mathbf{A}$ that witness M.

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