Critical points for congruence lattices

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General problem

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
(Con $\mathcal{K} \subseteq$ Con $\mathcal{L}$)

and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

(L_c = compact elements of L)

Why L_c?

$\mathrm{Con}_c A$ reflects the size of A better.

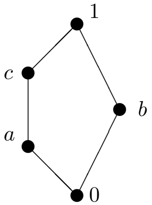
Theorem

If an infinite algebra A is a subdirect product of finite algebras of bounded size, then $|\mathrm{Con}_c A| = |A|$.

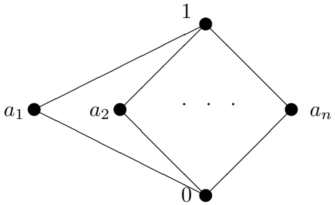
Some critical points

We are especially interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are congruence-distributive varieties (in most results also finitely generated). For instance,
$\mathrm{Crit}(\mathbf{N}_5, \mathbf{M}_3) = 5$,
$\mathrm{Crit}(\mathbf{M}_3, \mathbf{N}_5) = \mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) = \aleph_0$,
$\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$,
$\mathrm{Crit}(\mathbf{Maj}, \mathbf{Lat}) = \aleph_2$.
($\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{M}_4$, $\mathbf{D}$ are well-known lattice varieties, $\mathbf{Lat}$ = all lattices, $\mathbf{Maj}$ = all majority algebras.)
P. Gillibert: under some reasonable finiteness conditions, the critical point between two varieties cannot be larger than $\aleph_2$.

N_5 and M_n



N_5



M_n

No $\aleph_3$?

Theorem

(Gillibert)
Let $\mathcal{V}$ and $\mathcal{W}$ be locally finite varieties. Assume that for any finite $A \in \mathcal{V}$ there are, up to isomorphism, finitely many $B \in \mathcal{W}$ with $\mathrm{Con}_c\, A \cong \mathrm{Con}_c\, B$, and each such B is finite. Then $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_2$ or $\mathrm{Con}\,\mathcal{V} \subseteq \mathrm{Con}\,\mathcal{W}$.

Any finitely generated congruence-distributive varieties satisfy the assumptions.

Possible cases

For finitely generated congruence-distributive varieties there are following possible cases:

- $\mathrm{Crit}(\mathcal{V}, \mathcal{W})$ is finite;
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) = \aleph_0$;
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) = \aleph_1$;
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) = \aleph_2$;
- $\mathrm{Con}\,\mathcal{V} \subseteq \mathrm{Con}\,\mathcal{W}$.

How to distinguish?

Con functor

The Con functor:

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}\, f: \mathrm{Con}\, A \to \mathrm{Con}\, B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}\, f$ preserves $\vee$ and 0, not necessarily $\wedge$.

Lifting of semilattice morphisms

Let

- $\varphi: \; S \to T$ be a $(\vee, 0)$-homomorphisms of lattices;
- $f: \; A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1: \; S \to \operatorname{Con} A$, $\psi_2: \; T \to \operatorname{Con} B$ such that

$$\begin{array}{ccc} \operatorname{Con} A & \xrightarrow{\operatorname{Con} f} & \operatorname{Con} B \\ \psi_1 \big\downarrow & & \psi_2 \big\downarrow \\ S & \xrightarrow{\varphi} & T \end{array}$$

commutes.

A generalization: lifting of semilattice diagrams

$\aleph_0$ and $\aleph_1$ criteria

Theorem

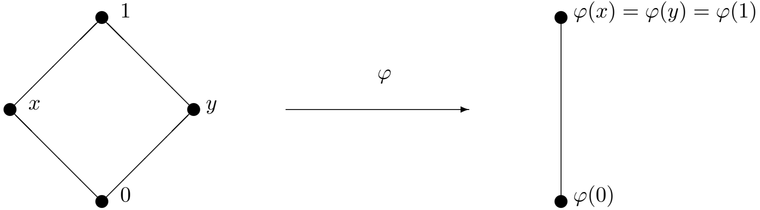
(Gillibert)
$\mathrm{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_0$ *if and only if there is a diagram of finite semilattices indexed by a finite chain, which is liftable in* $\mathcal{V}$ *but not in* $\mathcal{W}$.

Theorem

(Gillibert)
If there is a diagram of finite semilattices indexed by the product of two finite chains, which is liftable in $\mathcal{V}$ *but not in* $\mathcal{W}$*, then* $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_1$.

Example

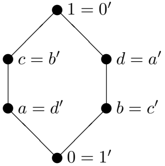
The semilattice homomorphism



has a lifting in $\mathbf{M}_3$ (the embedding of a 3-element chain into M_3 lifts it), but not in $\mathbf{D}$. Therefore, $\mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) \leq \aleph_0$.

Critical point $\aleph_1$

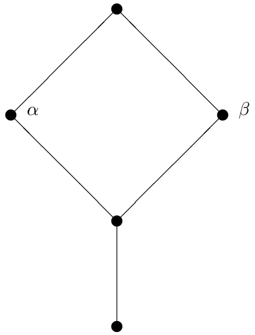
Let $\mathbf{N}_6^*$ be the varieties of bounded lattices with an additional unary operation $'$ generated by the algebra N_6.



N_6

Lattice T

N_5 and N_6^* have the same congruence lattice T:



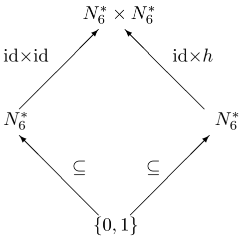
N6 versus N5

Let φ be the identity homomorphism $T \to T$. Now:

- There is only one possibility of lifting of φ in $\mathbf{N}_5$ (the identity $N_5 \to N_5$).
- There are two different liftings of φ in $\mathbf{N}_6^*$: the identity on N_6^* and the vertical symmetry $h : N_6^* \to N_6^*$. The homomorphism $\mathrm{Con}_c\, h$ interchanges α and β.

Critical points $\aleph_1$

For the following diagram $\mathcal{D}$ in $\mathbf{N}_6^*$, the corresponding semilattice diagram Con $\mathcal{D}$ has no lifting in $\mathbf{N}_5$. By Gillibert's criterion, $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{N}_5) \leq \aleph_1$.



Symmetries of liftings

Let $\varphi\colon\ S \to T$ be a homomorphism of $(\vee, 0)$-semilattices and let τ be an automorphism of T. A τ-*symmetric lifting* of φ in a variety $\mathcal{V}$ consists of algebras $A_1, A_2, B_1, B_2 \in \mathcal{V}$, homomorphisms $f_{ij}\colon\ A_i \to B_j$, isomorphisms $\psi_i\colon\ \operatorname{Con}_c A_i \to S$ and $\tau_{ij}\colon\ \operatorname{Con} B_j \to T$ such that

$$(\operatorname{rng} f_{11} \times f_{12}) \cap (\operatorname{rng} f_{21} \times f_{22}) \neq \emptyset,$$

the diagram

$$\begin{array}{ccc} \operatorname{Con} A_i & \xrightarrow{\operatorname{Con} f_{ij}} & \operatorname{Con} B_j \\ \psi_i \downarrow & & \tau_{ij} \downarrow \\ S & \xrightarrow{\varphi} & T \end{array}$$

commutes for every $i, j \in \{1, 2\}$, and

$$\tau = \tau_{11}\tau_{21}^{-1}\tau_{22}\tau_{12}^{-1}.$$

Main result

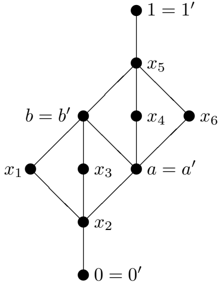
Theorem

Let $\varphi : S \to T$ be a homomorphism of finite $(\vee, 0)$-semilattices and let τ be an automorphism of T. If φ has a τ-symmetric lifting in $\mathcal{V}$ but not in $\mathcal{W}$, then $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_1$.

In the $\mathbf{N}_5$ versus $\mathbf{N}_6^*$ case we used the identity $\varphi : T \to T$ and the automorphism $\tau : T \to T$ interchanging α and β.

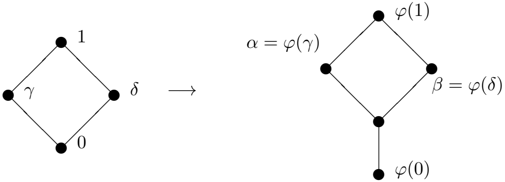
Another example

Let $\mathbf{M}^*$ be the variety of bounded lattices with an additional unary operation $'$ generated by the algebra M. depicted below. The unary operation on the elements x_i is defined by $x_i' = x_{i+1}$ and $x_6' = x_1$.



M

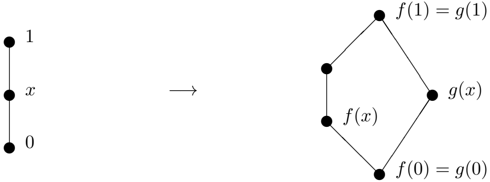
Let φ be the following semilattice homomorphism $S \to T$:



Let $\tau : T \to T$ be the same as before (interchanging α and β). Then φ has a τ-symmetric lifting in $\mathbf{N}_5$ but not in $\mathbf{M}$.

Lifting in N5

There is no τ-symmetric lifting of φ in $\mathbf{M}$. On the other hand, a τ-symmetric lifting of φ in $\mathbf{N}_5$ can be constructed using the following two embeddings $C_3 \to N_5$.



Possible generalizations

- G-symmetric lifings with G a subgroup of Aut(T);
- τ-symmetric (or G-symmetric) liftings of diagrams indexed by finite chains;

Thanks

Thank you for attention.

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