Profinite algebras and polynomial boundedness

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Profinite algebras

Profinite algebras and Stone spaces

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Question

Is this true for general topological algebras?

A counterexample

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Consider the Stone-Čech compactification $\beta\mathbb{N}$ of the natural numbers and let $f\colon \beta\mathbb{N} \to \beta\mathbb{N}$ be the continuous extension of the map $s\colon \mathbb{N} \to \mathbb{N}$, $n \mapsto n+1$.

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The map $\mathrm{Con}_o(\mathbf{A}) \to \mathrm{Con}(\mathbb{N}, s)$, $\theta \mapsto \theta \cap \mathbb{N}^2$ is injective.

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Proof.

The map $\mathrm{Con}_o(\mathbf{A}) \to \mathrm{Con}(\mathbb{N}, s)$, $\theta \mapsto \theta \cap \mathbb{N}^2$ is injective. Hence, $\mathrm{Con}_o(\mathbf{A})$ is countable. If $\mathbf{A}$ was profinite, then

$$\beta\mathbb{N} \to \prod_{\theta \in \mathrm{Con}_o(\mathbf{A})} \beta\mathbb{N}/\theta, \qquad a \mapsto ([a]_\theta)_{\theta \in \mathrm{Con}_o(\mathbf{A})}$$

would be an embedding, and thus $\beta\mathbb{N}$ would be metrizable. Contradiction! □

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Let Ω be a signature. For an Ω-algebra $\mathbf{A}$ let $M(\mathbf{A})$ denote the translation monoid of $\mathbf{A}$, i.e., the monoid generated by the maps

$$A \to A, \qquad x \mapsto \omega^{\mathbf{A}}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

where $n \in \mathbb{N} \setminus \{0\}$, $\omega \in \Omega_n$, $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$.

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A topological Ω-algebra $\mathbf{A}$ is profinite if and only if the following hold:

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Question

How can we assure the second condition?

Revisiting classical examples: groups

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If $\mathbf{G} = (G, \cdot, {}^{-1}, e)$ is a group, then

$$M(\mathbf{G}) = \{x \mapsto axb \mid a, b \in G\} \cup \{x \mapsto ax^{-1}b \mid a, b \in G\}.$$

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If $\mathbf{G}$ is a topological group, then the maps

$$\varphi_1 \colon G^2 \to M(\mathbf{G}),\ (a, b) \mapsto \lambda_a \circ \rho_b,$$
$$\varphi_2 \colon G^2 \to M(\mathbf{G}),\ (a, b) \mapsto \lambda_a \circ \rho_b \circ \iota$$

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are continuous w.r.t. the compact-open topology. Hence:

$$G \text{ compact} \implies M(\mathbf{G}) = \varphi_1(G^2) \cup \psi_2(G^2) \text{ compact.}$$

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If $\mathbf{R} = (R, +, \cdot)$ is a ring, then

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Polynomial boundedness

Representing the translation monoid

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Let Ω be a signature and A a set. Consider the extended signature $\Omega + A$ where $(\Omega + A)_0 := \Omega_0 + A$ and $(\Omega + A)_n := \Omega_n$ for $n \geq 1$. Let

$$L_{\Omega+A}(x) := \{t \in T_{\Omega+A}(x) \mid x \text{ occurs exactly once in } t\}.$$

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If $\mathbf{A}$ is an Ω-algebra, then we have $M(\mathbf{A}) = \psi_{\mathbf{A}}(L_{\Omega+A}(x))$ with regard to the canonical map $\psi_{\mathbf{A}} \colon T_{\Omega+A}(x) \to A^A$.

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Definition

An Ω-algebra $\mathbf{A}$ is called bounded if there exists some $m \in \mathbb{N}$ such that

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Groups, semigroups, rings, and distributive lattices are bounded!

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Proposition

If $\mathbf{A}$ is bounded and A is compact, then $M(\mathbf{A})$ is compact w.r.t. the compact-open topology.

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Sketch of proof.

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(1) Endow $T_{\Omega+A}(x)$ with a topology such that $\psi_{\mathbf{A}}\colon T_{\Omega+A}(x) \to C(A,A)$ is continuous w.r.t. the compact-open topology.

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(3) Conclude that $M(\mathbf{A}) = \psi_{\mathbf{A}}(L)$ is compact.

□

Bounded profinite algebras

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Theorem

Let Ω be a finite (compact) signature. If $\mathbf{A}$ is a bounded topological Ω-algebra, then the following are equivalent:

(1) $\mathbf{A}$ *is profinite.*

(2) A *is a Stone space.*

Thanks!