Permutation Pattern Avoidance and Constraint Satisfaction Problems

AAA90 : Novi Sad, Serbia

Tom Hanika

Universität Kassel

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- Structural properties leading to an algorithm for $P = \{231\}$.

Permutations etc.

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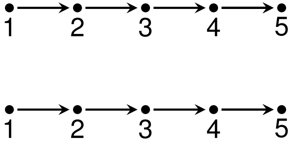

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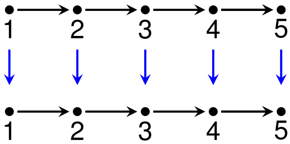

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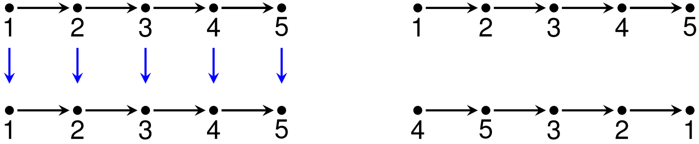

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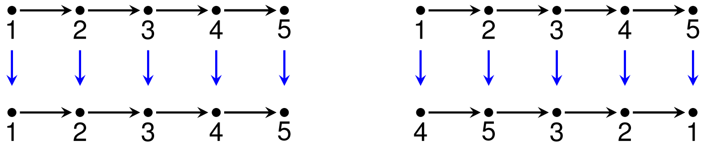

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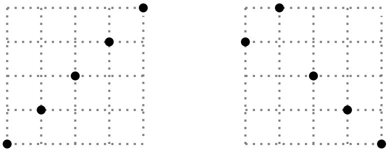

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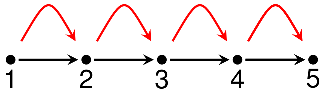

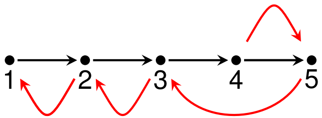

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One permutation to another

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$\sigma = s_1 s_2 s_3 \dots s_k$ is sequence $:\Leftrightarrow$ s_i distinct numbers taken from $(\mathbb{N}, \leq_\mathbb{N})$.
Sequences $\sigma = s_1 s_2 \dots s_k$, $\tau = t_1 t_2 \dots t_k$ are order isomorphic ($\sigma \cong \tau$), if

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Given $\pi = 14352$, then

- $\sigma = 132$ is involved in π, since 142 is a subsequence of π.

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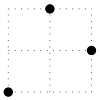
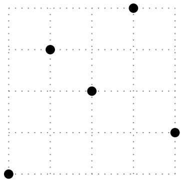
132  $\preceq$  14352

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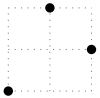
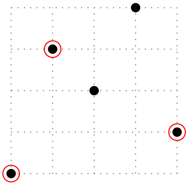
132  $\preceq$  14352

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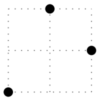
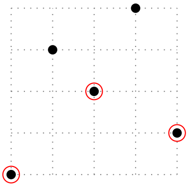
132  $\preceq$  14352

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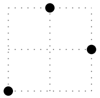
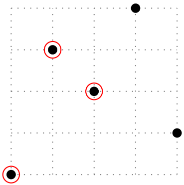
132  $\preceq$  14352

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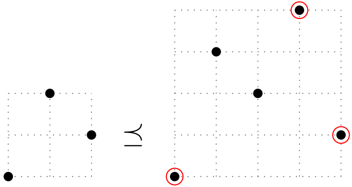

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Definition

Let $\pi \in S_n$ and τ a permutation (pattern) of length k with $k \leq n$. The permutation π avoids τ if $\tau \not\preceq \pi$. Let T be a set of patterns.

$$\mathcal{A}_n(T) := \{\sigma \in S_n \mid \tau \not\preceq \sigma \text{ for all } \tau \in T\}$$

Separable and stack-sortable permutations

Direct sum and skew sum

Definition

For $\pi \in S_n$ and $\tau \in S_m$, the direct sum of π and τ is defined by

$$\pi \oplus \tau := \pi(1), \ldots, \pi(n), \tau(1) + n, \ldots, \tau(m) + n,$$

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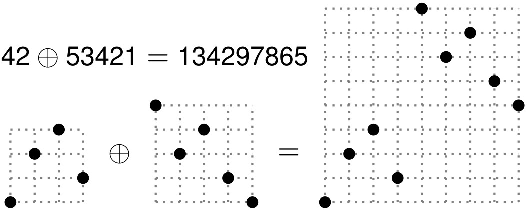

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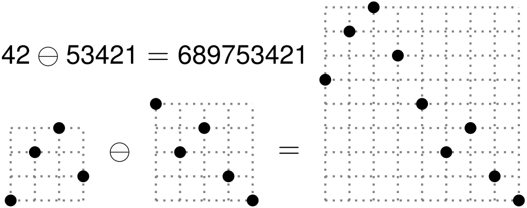

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Theorem (Folklore)

Permutations that do not involve 2413 *and* 3142 *are separable, i. e.,* $\mathcal{A}(3124, 2413)$.

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- $\sigma \oplus \tau \in \mathcal{A}_{\leq n}(231)$
- $\sigma \ominus \tau \in \mathcal{A}_{\leq n}(231)$, *iff σ is not ascending.*

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Input: *A tuple* $(\underline{n}, E_1, E_2)$ *where* $\underline{n} = \{1, \ldots, n\}$*, and* E_1, E_2 *are binary relations on* $\underline{n}$*.*

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Theorem

FPCSP($\{231\}$) *is solvable in polynomial time.*

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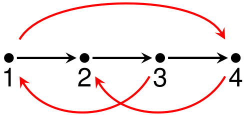
- *$G := (V, E_1 \cup E_2)$ has more than one scc, or*
- *$G' := (V, E_1 \cup E_2^{-1})$ has more than one strongly connected component and a final scc C has no accents (i.e., there is a solution to FPCSP(12)).*

Outline of algorithm by example

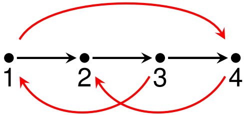
Example 1



Outline of algorithm by example

1. Identify strongly connected components

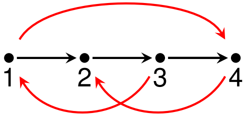
Example 1



Outline of algorithm by example

1. Identify strongly connected components
2. If only one, check inverted with E_2

Example 1



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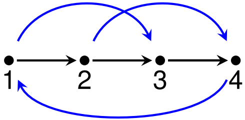

Outline of algorithm by example

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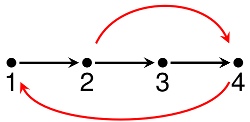
Example 2



Outline of algorithm by example

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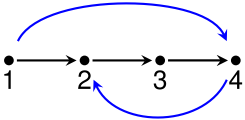
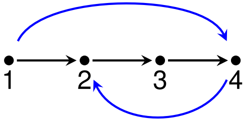
Example 2



Outline of algorithm by example

1. Identify strongly connected components
2. If only one, check inverted with E_2
3. If more than one scc, check a final scc for not accending calls on terminal scc and complement.

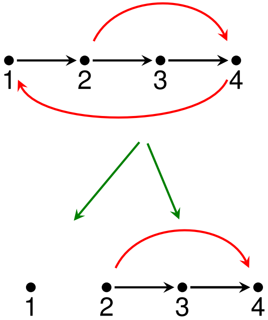
Example 2



Outline of algorithm by example

1. Identify strongly connected components
2. If only one, check inverted with E_2
3. If more than one scc, check a final scc for not accending calls on terminal scc and complement.

Example 2



Thank you very much!