Factor Congruence Lifting Property

IN RELATION TO OTHER LIFTING PROPERTIES

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Research on lifting properties up to now

Previous research on lifting properties:

- in **ring theory**: *Idempotent Lifting Property (LIP)*, extensively studied
- in **MV–algebras** and **BL–algebras** (algebras of many–valued logics): the *Boolean Lifting Property* modulo the radical (the intersection of all maximal filters)

Work on lifting properties done by Professor George Georgescu and me, later joined by Daniela Cheptea:

- *Boolean Lifting Property* in **residuated lattices** (a wider class of algebras of many–valued logics), first modulo the radical, then modulo all filters
- a generalization to **universal algebras**: the *φ–lifting properties*, with φ a formula with one free variable, logically equivalent to a positive formula, from the first order language associated to a type of universal algebras
- lifting properties obtained as particularizations of the φ–lifting properties: the *Idempotent Lifting Property* in **residuated lattices** (the *Boolean Lifting Property* can also be obtained as such a particularization) and the *Boolean Lifting Property* in **bounded distributive lattices** (three important cases of the latter: modulo all congruences, modulo the congruences associated to ideals and modulo the congruences associated to filters)
- other generalizations to **universal algebras**: the *Congruence Boolean Lifting Property*, and now the *Factor Congruence Lifting Property*

Some notations and basic notions

Throughout this presentation, any algebra shall be designated by its support set and shall be considerred non–empty.

- A : an algebra
- $(\mathrm{Con}(A), \vee, \cap, \Delta_A, \nabla_A)$: the bounded lattice of congruences of A
- A : *congruence–distributive* iff the lattice $\mathrm{Con}(A)$ is distributive
- A : *congruence–permutable* iff each $\phi, \psi \in \mathrm{Con}(A)$ satisfy $\phi \circ \psi = \psi \circ \phi$
- A : *arithmetical* iff it is both congruence–distributive and congruence–permutable
- let this be *condition (H):* ∇_A is a compact element of $\mathrm{Con}(A)$ (equivalently, ∇_A is a finitely generated congruence of A)
- if A fulfills (H), then all its maximal congruences are prime congruences

- Bounded distributive lattices fulfill (H) and they are congruence–distributive, but not necessarily congruence–permutable. Clearly, they form an equational class.
- L : a bounded distributive lattice
- the *Boolean center* of L: $\mathcal{B}(L) =$ the set of the complemented elements of L
- $\mathcal{B}(L)$ is a Boolean sublattice of L

Some notations and basic notions

Definition

(Commutative) residuated lattice: algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$, where:

- $(A, \vee, \wedge, 0, 1)$: bounded lattice (with partial order $\leq$)
- $(A, \odot, 1)$: commutative monoid
- $\rightarrow$: binary operation on A which satisfies the *law of residuation:* for all $a, b, c \in A$, $a \leq b \rightarrow c$ iff $a \odot b \leq c$

- Residuated lattices are arithmetical algebras and fulfill (H). They form an equational class.
- A : a residuated lattice
- then $(A, \vee, \wedge, 0, 1)$ is not necessarily distributive, but it is uniquely complemented
- the *Boolean center* of A: $\mathcal{B}(A) =$ the set of the complemented elements of $(A, \vee, \wedge, 0, 1)$
- $\mathcal{B}(A)$ is a Boolean sublattice of $(A, \vee, \wedge, 0, 1)$

Some notations and basic notions

- R, Q : residuated lattices or bounded distributive lattices
- $f : R \to Q$ a residuated lattice morphism or a bounded lattice morphism
- then $f(\mathcal{B}(R)) \subseteq \mathcal{B}(Q)$
- $\mathcal{B}(f) : \mathcal{B}(R) \to \mathcal{B}(Q)$, $\mathcal{B}(f) = f \mid_{\mathcal{B}(R)}$
- then $\mathcal{B}(f)$ is a Boolean morphism
- thus $\mathcal{B}$ becomes a covariant functor from the category of residuated lattices (or bounded distributive lattices – same notation) to that of Boolean algebras

- $\mathcal{L}$ = the *reticulation functor*: a covariant functor from the category of residuated lattices to that of bounded distributive lattices with "good" preservation properties, which make it perfect for transferring algebraic and topological properties from bounded distributive lattices to residuated lattices
- if A is a residuated lattice, then $\mathcal{L}(A)$ is isomorphic to the bounded distributive lattice of the principal filters of A and has the prime spectrum (that is the set of prime filters), endowed with the Stone topology, homeomorphic to that of A; this topological property is satisfied by only one bounded distributive lattice, modulo a bounded lattice isomorphism

More notations and basic notions

- A: a residuated lattice
- *filter* of A: subset $\emptyset \neq F \subseteq A$ such that all $a, b \in A$ satisfy:
  1. if $a, b \in F$, then $a \odot b \in F$
  2. if $a \in F$ and $a \leq b$, then $b \in F$
- $(\mathrm{Filt}(A), \vee, \cap, \{1\}, A) =$ the bounded distributive lattice of filters of A
- the bounded distributive lattices $\mathrm{Filt}(A)$ and $\mathrm{Con}(A)$ are isomorphic (if A would be a bounded distributive lattice, then we would only have an embedding here, not an isomorphism)
- $F \in \mathrm{Filt}(A)$; $\sim_F \in \mathrm{Con}(A)$: the image of F through the isomorphism mentioned above; notations: for any $a \in A$ and any $X \subseteq A$: $a/F = a/\sim_F$ and $X/F = X/\sim_F$; $p_F : A \rightarrow A/F$: the canonical surjective morphism

- A : an algebra
- $\theta \in \mathrm{Con}(A)$
- $p_\theta : A \rightarrow A/\theta$: the canonical surjective morphism

Boolean Lifting Property (BLP)

Definition

- A : a residuated lattice
- $F \in \mathrm{Filt}(A)$
- $\sim_F \in \mathrm{Con}(A)$: the congruence associated to F
- F has the *Boolean Lifting Property (BLP)* iff $\mathcal{B}(A/F) = \mathcal{B}(A)/F$ (equivalently: iff the Boolean morphism $\mathcal{B}(p_F) : \mathcal{B}(A) \to \mathcal{B}(A/F)$ is surjective)
- $\sim_F$ has the *BLP* iff F has the BLP
- A has the *BLP* iff all its filters have the BLP (equivalently: iff all its congruences have the BLP)

Definition

- L : a bounded distributive lattice
- $\theta \in \mathrm{Con}(L)$
- θ has the *Boolean Lifting Property (BLP)* iff $\mathcal{B}(L/\theta) = \mathcal{B}(L)/\theta$ (equivalently: iff the Boolean morphism $\mathcal{B}(p_\theta) : \mathcal{B}(L) \to \mathcal{B}(L/\theta)$ is surjective)
- L has the *BLP* iff all its congruences have the BLP

Congruence Boolean Lifting Property (CBLP)

- A : a congruence–distributive algebra
- $\theta \in \mathrm{Con}(A)$
- $s_\theta : \mathrm{Con}(A/\theta) \rightarrow [\theta) = \{\alpha \in \mathrm{Con}(A) \mid \theta \subseteq \alpha\}$, for all $\psi \in \mathrm{Con}(A/\theta)$, $s_\theta(\psi) = \{(a, b) \mid a, b \in A, (a/\theta, b/\theta) \in \psi\}$
- s_θ is a bounded lattice isomorphism
- hence $\mathcal{B}(s_\theta)$ is a Boolean isomorphism
- $u_\theta : \mathrm{Con}(A) \rightarrow \mathrm{Con}(A/\theta)$, for all $\psi \in \mathrm{Con}(A)$, $u_\theta(\psi) = (\psi \vee \theta)/\theta$
- $v_\theta : \mathrm{Con}(A) \rightarrow [\theta)$, for all $\psi \in \mathrm{Con}(A)$, $v_\theta(\psi) = \psi \vee \theta$
- the following diagrams are commutative:

$$\begin{tikzcd} \mathrm{Con}(A) \arrow[rr, "u_\theta"] \arrow[dr, "v_\theta"'] & & \mathrm{Con}(A/\theta) \arrow[dl, "s_\theta"] \\ & {[\theta)} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}(\mathrm{Con}(A)) \arrow[rr, "\mathcal{B}(u_\theta)"] \arrow[dr, "\mathcal{B}(v_\theta)"'] & & \mathcal{B}(\mathrm{Con}(A/\theta)) \arrow[dl, "\mathcal{B}(s_\theta)"] \\ & \mathcal{B}([\theta)) & \end{tikzcd}$$

Definition

With the notations above:

- θ has the *Congruence Boolean Lifting Property (CBLP)* iff $\mathcal{B}(u_\theta)$ is surjective (equivalently, iff $\mathcal{B}(v_\theta)$ is surjective)
- A has the *CBLP* iff all its congruences have the CBLP

Factor Congruence Lifting Property (FCLP)

- A : a congruence–distributive algebra
- $\theta \in \mathrm{Con}(A)$
- θ : *factor congruence* iff there exists a $\theta^* \in \mathrm{Con}(A)$ such that $\theta \vee \theta^* = \nabla_A$, $\theta \cap \theta^* = \Delta_A$ and $\theta \circ \theta^* = \theta^* \circ \theta$ (that is iff $\theta \in \mathcal{B}(\mathrm{Con}(A))$ and θ permutes with its complement)
- $\mathcal{FC}(A) =$ the set of the factor congruences of A
- $\mathcal{FC}(A)$ is a Boolean subalgebra of $\mathcal{B}(\mathrm{Con}(A))$
- $u_\theta(\mathcal{FC}(A)) \subseteq \mathcal{FC}(A/\theta)$
- **notation:** $\mathcal{FC}(\theta) : \mathcal{FC}(A) \rightarrow \mathcal{FC}(A/\theta)$, $\mathcal{FC}(\theta) = u_\theta \mid_{\mathcal{FC}(A)}$
- $\mathcal{FC}(\theta)$ is a Boolean morphism

Definition

With the notations above:

- θ has the *Factor Congruence Lifting Property (FCLP)* iff $\mathcal{FC}(\theta)$ is surjective
- A has the *FCLP* iff all its congruences have the FCLP

BLP, CBLP and FCLP

We shall see how results on BLP in residuated lattices and bounded distributive lattices generalize to CBLP and FCLP in universal algebras, as well as some correlations between these lifting properties.

- All these lifting properties are preserved by quotients and finite direct products.
- If A is a congruence–distributive algebra, then Δ_A and ∇_A have CBLP and FCLP. The same holds for BLP if A is a residuated lattice or a bounded distributive lattice.
- Boolean algebras have BLP, CBLP and FCLP.
- Bounded distributive lattices have CBLP.
- All algebras fulfilling (H) from a discriminator equational class have CBLP.
- Examples of discriminator equational classes with all members fulfilling (H): Boolean algebras, Post algebras, n–valued MV–algebras, monadic algebras, cylindric algebras, Gödel residuated lattices.
- Well–known fact: discriminator equational classes are arithmetical, that is: all algebras from a discriminator equational class are arithmetical.

In arithmetical algebras, FCLP and CBLP coincide

Obviously, if A is an arithmetical algebra, then:

- $\mathcal{FC}(A) = \mathcal{B}(\mathrm{Con}(A))$
- so are all its quotient algebras

Thus, if A is an arithmetical algebra:

- any $\theta \in \mathrm{Con}(A)$ satisfies: θ has FCLP iff θ has CBLP
- A has FCLP iff A has CBLP

Consequently:

- all algebras fulfilling (H) from a discriminator equational class have FCLP;
- if A is a residuated lattice, then:
  any $\theta \in \mathrm{Con}(A)$ satisfies: θ has FCLP iff θ has CBLP
  A has FCLP iff A has CBLP

CBLP and FCLP versus BLP

- L: a bounded distributive lattice
- $\theta \in \mathrm{Con}(A)$
- θ has FCLP iff θ has BLP
- L has FCLP iff L has BLP
- As mentioned above, in bounded distributive lattices, CBLP always holds.

- A: a residuated lattice
- $F \in \mathrm{Filt}(A)$
- **definition:** F has *CBLP* iff $\sim_F$ has CBLP
- **definition:** F has *FCLP* iff $\sim_F$ has FCLP
- F has FCLP iff F has CBLP iff F has BLP
- A has FCLP iff A has CBLP iff A has BLP

Prime congruences have BLP, CBLP and FCLP

- Prime filters of residuated lattices have BLP, thus also CBLP and FCLP.
- Prime congruences of bounded distributive lattices have BLP, thus also FCLP.
- Prime congruences of congruence–distributive algebras fulfilling (H) have CBLP and FCLP.

Consequently:

- Maximal filters of residuated lattices have BLP, thus also CBLP and FCLP.
- Maximal congruences of bounded distributive lattices have BLP, thus also FCLP.
- Maximal congruences of congruence–distributive algebras fulfilling (H) have CBLP and FCLP.

Consequently:

- Local residuated lattices have BLP, thus also CBLP and FCLP.
- Local bounded distributive lattices have BLP, thus also FCLP.
- Local congruence–distributive algebras fulfilling (H) have CBLP and FCLP.

Furthermore: representation theorems

- Semilocal residuated lattices with BLP/CBLP/FCLP are exactly the finite direct products of local residuated lattices.
- Semilocal bounded distributive lattices with BLP/FCLP are exactly the finite direct products of local bounded distributive lattices.
- Semilocal algebras satisfying CBLP/FCLP from an arithmetical equational class with all members fulfilling (H) are exactly the finite direct products of local algebras from that class.

- Maximal residuated lattices with BLP/CBLP/FCLP are exactly the finite direct products of local maximal residuated lattices.
- Maximal bounded distributive lattices with BLP/FCLP are exactly the finite direct products of local maximal bounded distributive lattices.
- Maximal algebras satisfying CBLP/FCLP from an arithmetical equational class with all members fulfilling (H) are exactly the finite direct products of local maximal algebras from that class.

Some definitions

- L: a bounded distributive lattice
- L: *normal* iff, for all $x, y \in L$, if $x \vee y = 1$, then there exist $e, f \in L$ such that $e \wedge f = 0$ and $x \vee e = y \vee f = 1$
- L: *B–normal* iff, for all $x, y \in L$, if $x \vee y = 1$, then there exist $e, f \in \mathcal{B}(L)$ such that $e \wedge f = 0$ and $x \vee e = y \vee f = 1$

- A: a congruence–distributive algebra
- A: *B–normal* iff $\mathrm{Con}(A)$ is B–normal, that is iff, for all $\phi, \psi \in \mathrm{Con}(A)$, if $\phi \vee \psi = \nabla_A$, then there exist $\alpha, \beta \in \mathcal{B}(\mathrm{Con}(A))$ such that $\alpha \cap \beta = \Delta_A$ and $\phi \vee \alpha = \psi \vee \beta = \nabla_A$
- A: *FC–normal* iff, for all $\phi, \psi \in \mathrm{Con}(A)$, if $\phi \vee \psi = \nabla_A$, then there exist $\alpha, \beta \in \mathcal{FC}(\mathrm{Con}(A))$ such that $\alpha \cap \beta = \Delta_A$ and $\phi \vee \alpha = \psi \vee \beta = \nabla_A$

- Clearly, if A is arithmetical, then: A is B–normal iff A is FC–normal.

Characterizations for CBLP and FCLP

- A: a congruence–distributive algebra

If A fulfills (H), then the following are equivalent:

- A has CBLP
- A is B–normal
- the set of the prime congruences of A, endowed with a topology defined analogously to the Stone topology for residuated lattices, is a strongly zero–dimensional topological space; let's call this *property* τ

The following are equivalent:

- A has FCLP
- A is FC–normal

Consequently, if A is an arithmetical algebra fulfilling (H), then the following are equivalent:

- A has CBLP
- A has FCLP
- A is B–normal
- A is FC–normal
- property τ

- A: a residuated lattice

Then the following are equivalent:

- A has FCLP
- A has CBLP
- A has BLP
- $\mathcal{L}(A)$ has BLP
- $\mathcal{L}(A)$ has FCLP
- $\mathrm{Filt}(A)$ is normal
- the dual of $\mathcal{L}(A)$ is normal
- any prime filter of A is included in a unique maximal filter of A
- any prime filter of $\mathcal{L}(A)$ is included in a unique maximal filter of A
- the set of prime filters of A, endowed with the Stone topology, is a normal topological space
- the set of prime filters of $\mathcal{L}(A)$, endowed with the Stone topology, is a normal topological space

So the functor $\mathcal{L}$ preserves and reflects the BLP and the FCLP. Trivially, $\mathcal{L}$ preserves the CBLP; $\mathcal{L}$ does not reflect the CBLP. Note that the last phrase in the abstract I have submitted is inexact.

Are CBLP and FCLP independent?

- L: a bounded distributive lattice

Definition

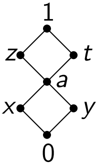
- L has Filt–BLP iff all its congruences associated to filters have BLP
- L has Id–BLP iff all its congruences associated to ideals have BLP

The following equivalences hold:

- L has Filt–BLP iff its lattice of filters is B–normal
- L has Id–BLP iff its lattice of ideals is B–normal
- as we have seen: L has BLP iff L has FCLP

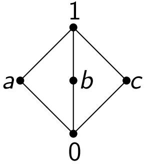
The following show that **CBLP does not imply FCLP**:

- as mentioned above, all bounded distributive lattices have CBLP
- not all bounded distributive lattices have BLP, thus not all bounded distributive lattices have FCLP; for instance, the following bounded distributive lattice doesn‘t even have Filt–BLP or Id–BLP:

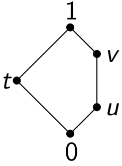


Things are different in non–distributive bounded lattices

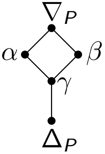
Let us illustrate the definitions for the CBLP and the FCLP by two **examples**: in the smallest non–distributive lattices.
The diamond has CBLP and FCLP. This is trivial, since its lattice of congruences is the two–element chain: $\mathrm{Con}(D) = \{\Delta_D, \nabla_D\}$.



diamond (D)



pentagon (P)

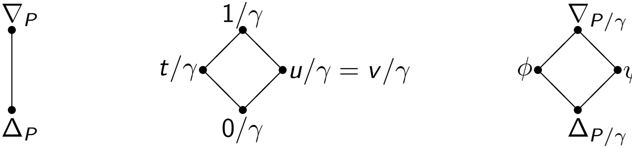


$\mathrm{Con}(P)$

The pentagon has neither CBLP, nor FCLP. In the diagram above of its lattice of congruences, α, β and γ correspond to the partitions $\{\{0, u, v\}, \{t, 1\}\}$, $\{\{0, t\}, \{u, v, 1\}\}$ and $\{\{0\}, \{t\}, \{u, v\}, \{1\}\}$, respectively.
Let us verify that γ has neither CBLP, nor FCLP.
Clearly, $\mathcal{B}(\mathrm{Con}(P)) = \{\Delta_P, \nabla_P\} = \mathcal{FC}(P)$.

Things are different in non–distributive bounded lattices

P/γ is a Boolean algebra, hence, since it's finite, it is isomorphic to its lattice of congruences; in the diagram of $\mathrm{Con}(P/\gamma)$ below, ϕ and ψ correspond to the partitions $\{\{0/\gamma, t/\gamma\}, \{u/\gamma, 1/\gamma\}\}$ and $\{\{0/\gamma, u/\gamma\}, \{t/\gamma, 1/\gamma\}\}$, respectively. Notice that $\mathrm{Con}(P/\gamma) = \mathcal{B}(\mathrm{Con}(P/\gamma)) = \mathcal{FC}(P/\gamma)$.



$\mathcal{B}(\mathrm{Con}(P)) = \mathcal{FC}(P)$ $\qquad$ P/γ $\qquad$ $\mathrm{Con}(P/\gamma) = \mathcal{B}(\mathrm{Con}(P/\gamma)) = \mathcal{FC}(P/\gamma)$

Hence the following Boolean morphisms are defined on the two–element Boolean algebra, with values in the four–element Boolean algebra, thus neither of them is surjective:

$$\mathcal{B}(u_\gamma) : \mathcal{B}(\mathrm{Con}(P)) \to \mathcal{B}(\mathrm{Con}(P/\gamma))$$

$$\mathcal{FC}(\gamma) : \mathcal{FC}(P) \to \mathcal{FC}(P/\gamma)$$

Thus γ has neither CBLP, nor FCLP. Therefore P has neither CBLP, nor FCLP.

THANK YOU FOR YOUR ATTENTION!