Cube Term Blockers Without Finiteness

Ágnes Szendrei

CU Boulder/U Szeged
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Joint work with Keith Kearnes

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- $\diamond$ for finite $\mathbf{A}$, implies that $\mathbf{A}$ is finitely related [Aichinger, Mayr, McKenzie, 2014]

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- ◇ **Note.** (2) $\Rightarrow$ (1) is easy.

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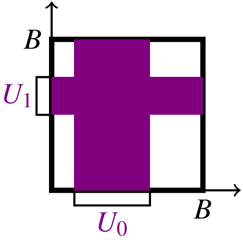

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Easy Fact: Let $\mathbf{A}$ be an idempotent algebra. TFAE for $\mathbf{B} \leq \mathbf{A}$ and $\emptyset \subsetneq U \subsetneq B$:

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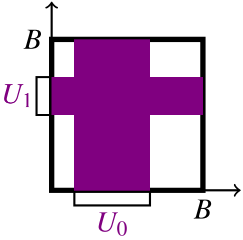

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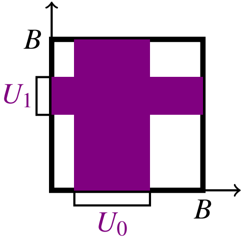

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- ◇ No finiteness assumption on $\mathcal{V}$; works for finite ν as well.
- ◇ But: the cross sequence is not constant.

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Corollary. *TFAE for any finite idempotent algebra* $\mathbf{A}$*:*

1. $\mathbf{A}$ *has no cube term;*
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- ⋄ (1) $\Leftrightarrow$ (4) is the MMM Theorem
- ⋄ Reason for $\neg$(4) $\Rightarrow$ $\neg$(3):
  In any signature, the class of idempotent algebras with no cube term blockers is closed under H, S, and P_{fin}.

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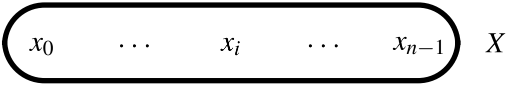

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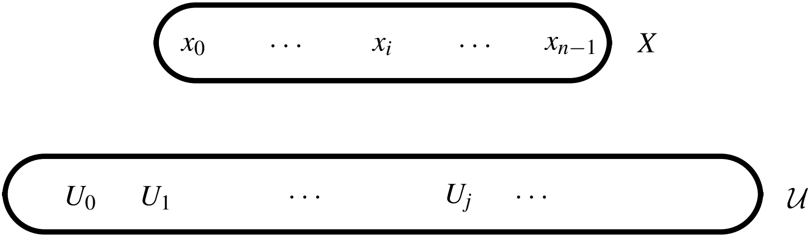

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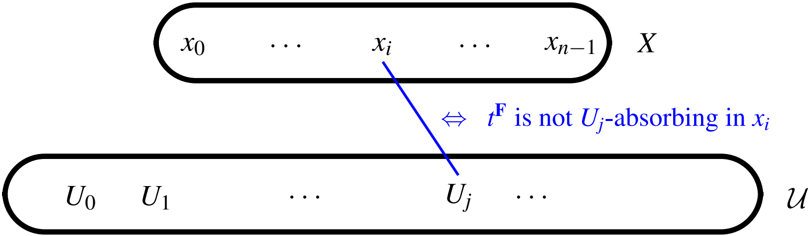

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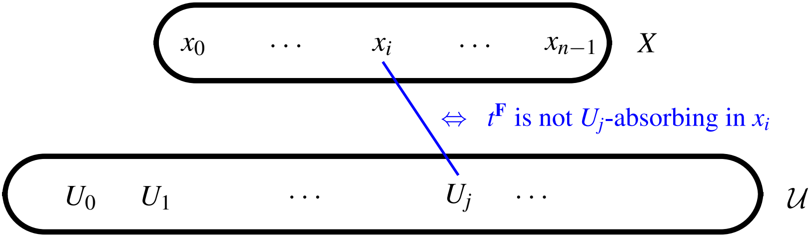

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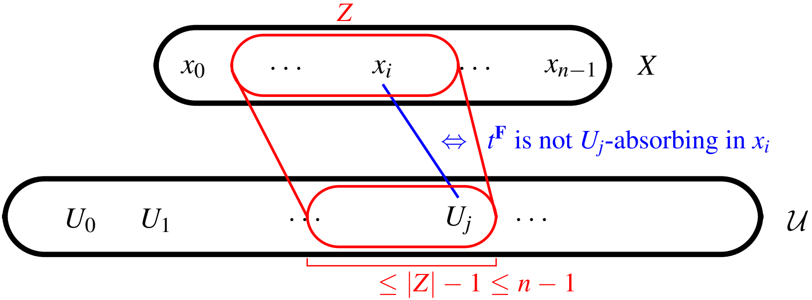

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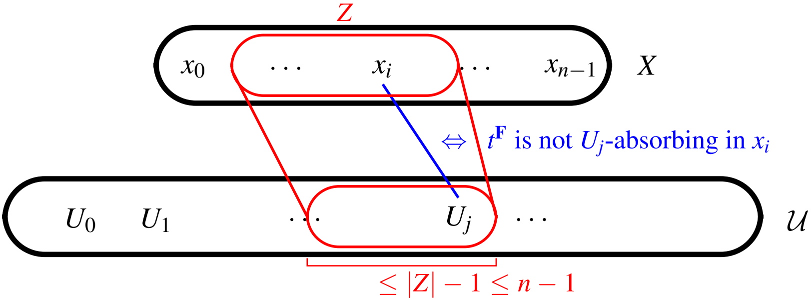

  - ◇ There is no matching from X to $\mathcal{U}$.
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Proof of Thm 2: Symmetrizing Infinite Cross Sequences

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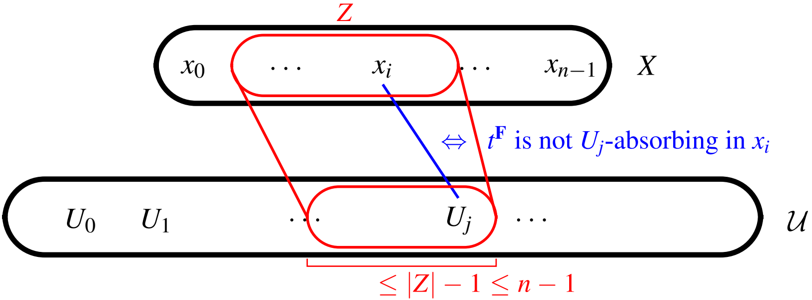

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Corollary. *Assume $\mathcal{V}$ has one operation symbol only, which is binary. Then either $\mathcal{V}$ has a Mal'tsev term, or it has no cube term at all.*

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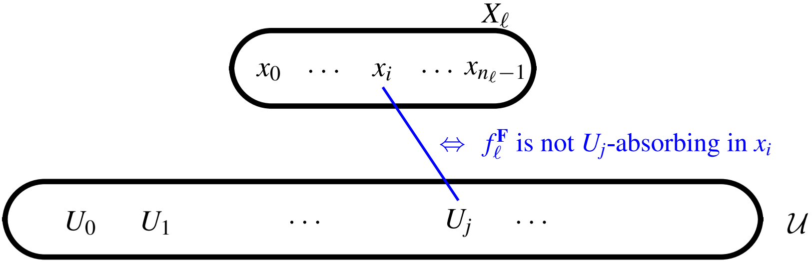

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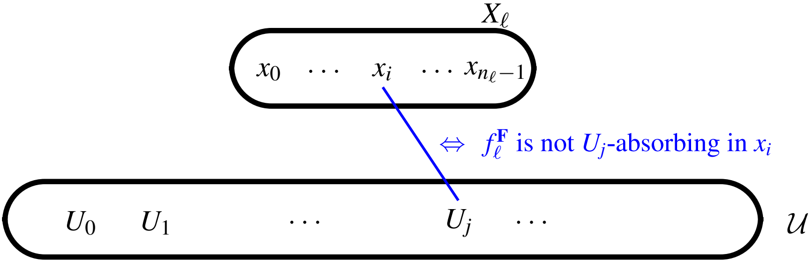

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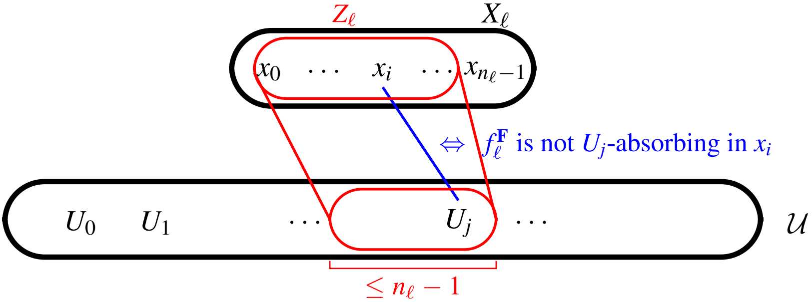

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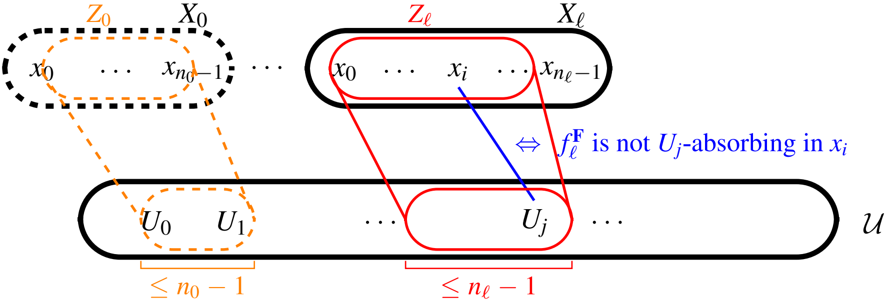

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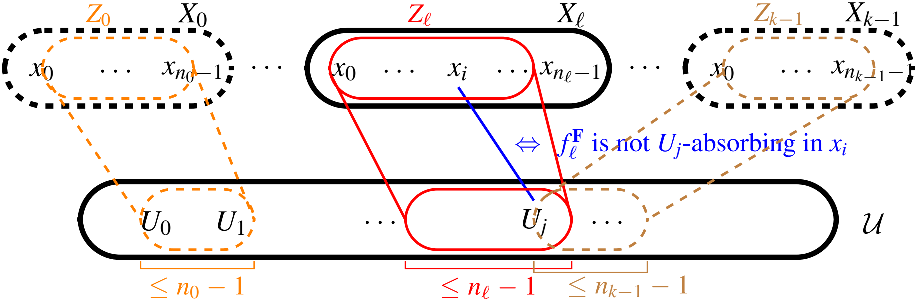

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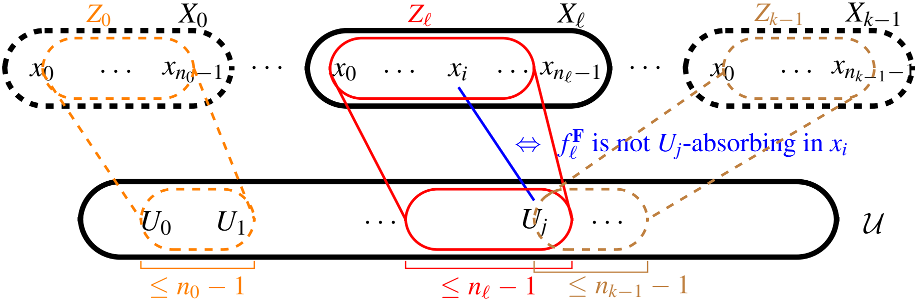

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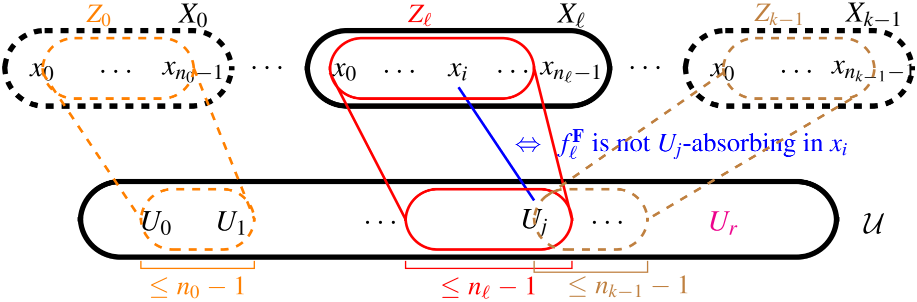

- ⋄ There is no matching from X_ℓ to $\mathcal{U}$.
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- ⋄ $d > \sum_{\ell<k}(n_\ell - 1) \Rightarrow \exists\, U_r \in \mathcal{U}$ s.t. all $f_\ell^{\mathbf{F}}$ have U_r-absorbing variables.