Topics in Hilbert algebras

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Abstract

Hilbert algebras are important tools for certain investigations in intuitionistic logic and other non-classical logics.

We introduce new types of filters in Hilbert algebra and study them in detail and put in evidence new connections between different types of filters.

After that, we introduce new types of Hilbert algebras and investigate their properties. Also, we give the relationships between these Hilbert algebras and other algebraic structures.

1 Introduction and Preliminaries

The variety of Hilbert algebras is an important tool for investigations in intuitionistic logic and other non-classical logics. Hilbert algebras represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. Hilbert algebra was studied by A. Diego[8] and D. Buşneag [1, 2, 3, 4], S. A. Celani, D. Montangie [7], S. A. Celani [5], S. M. Hong , Y. B. Jun[12] and Figallo et al.[9].

The aim of this work is to study filter theory in Hilbert algebras. The concepts positive implicative filters, Boolean filters and fantastic filters in Hilbert algebras are introduced and relationships between these filters and quotient algebras that are constructed via these filters are described.

The structure of the paper is as follows: in Section 2, some definitions and properties about Hilbert algebras are recalled. In Section 3, implicative filters are developed in Hilbert algebras. In Section 4, the notion positive implicative filter in Hilbert algebras is introduced, and we characterized them. In Section 5, the notion fantastic filter is defined in Hilbert algebra. We give relationships between fantastic filters and other filters. In Section

6, the notion Boolean filters in Hilbert algebras is developed and we prove that the positive implicative filters, fantastic filters and Boolean filters in bounded Hilbert algebras are equivalent. In Section 7, prime filters of the first kind, prime filters of the second kind and prime filters of the third kind are introduced and prime filters in Hilbert algebras are characterized.

2 Preliminaries

We recall some basic definitions and results that are necessary for this paper.

A Hilbert algebra is an algebra $(H, \to, 1)$ of type $(2, 0)$ such that the following axioms hold, for all $x, y, z \in H$:

$(H1)$ $x \to (y \to x) = 1$;

$(H2)$ $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

$(H3)$ if $x \to y = y \to x = 1$, then $x = y$.

It is proved that the above definition is equivalent to the system $\{H_4, H_5, H_6, H_7\}$, where:

$(H4)$ $x \to x = 1$;

$(H5)$ $1 \to x = x$;

$(H6)$ $x \to (y \to z) = (x \to y) \to (x \to z)$;

$(H7)$ $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

For a Hilbert algebra H, $(H, \leq)$ is a poset by defining an order relation $\leq$ as $x \leq y$ if and only if $x \to y = 1$. (called the natural order on H), with respect to this order, 1 is the greatest element of H. If H has a smallest element 0, we say that H is bounded, in this case we define a unary operation $*$ as $x^* = x \to 0$, for each $x \in H$.

A Hilbert algebra is prelinear if $(x \to y) \vee (y \to x) = 1$, for all $x, y \in H$. Linear (i.e., totally ordered) Hilbert algebras are prelinear. In linear Hilbert algebra with supremum, 1 is $\vee$- irreducible. This means that for all x and $y \in H$, $x \vee y = 1$ if and only if $x = 1$ or $y = 1$.

Example 2.1. [1]

(i) If $(H, \leq)$ is a poset with 1, then $(H, \to, 1)$ is a Hilbert algebra, where $x, y \in H$,

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x \nleq y \end{cases}$$

(*ii*) If $(H, \wedge, \vee, \to, 0)$ is a Heyting algebra, then $(H, \to, 1)$ is a Hilbert algebra.

(*iii*) There are Hilbert algebras which are not Heyting or Boolean algebras [8].

The following proposition include some properties of Hilbert algebra [1, 4, 8].

Proposition 2.2. *In each Hilbert algebra H, the following relations hold for all $x, y, z \in H$,*

(h1) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$;

(h2) $x \leq y \to x$, $x \leq (x \to y) \to y$;

(h3) $x \to (x \to y) = x \to y$;

(h4) $((x \to y) \to y) \to y = x \to y$;

(h5) $x \to (y \to z) = y \to (x \to z)$;

(h6) $x \to y \leq (y \to z) \to (x \to z)$;

(h7) $x \to y \leq (z \to x) \to (z \to y)$;

(h8) *If* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$;

(h9) $x \to (y \to z) = (x \to y) \to (x \to z)$;

(h10) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$;

(h11) $(x \to y) \to (y \to x) = y \to x$;

(h12) $((x \to y) \to x) \to y = x \to y$.

Proposition 2.3. *[4] If H is a bounded Hilbert algebra and $x, y \in H$, then*

(bh1) $0^* = 1$, $1^* = 0$;

(bh2) $x \to y^* = y \to x^*$;

(bh3) $x \to x^* = x^*$, $x^* \to x = x^{**}$, $x \le x^{**}$, $x \le x^* \to y$;

(bh4) $x \to y \le y^* \to x^*$;

(bh5) *If* $x \le y$, *then* $y^* \le x^*$;

(bh6) $x^{***} = x^*$;

(bh7) $(x \to y)^{**} = x \to y^{**} = x^{**} \to y^{**}$;

Definition 2.4. [9] A Hilbert algebra $(H, \to, 1)$ is called a Hilbert algebra with infimum when the underlying structure $(H, \le)$ with the order induced by $\to$ is a meet- semilattice.

Or, equivalently

Definition 2.5. [9] A Hilbert algebra with infimum is an algebra $(H, \to, \wedge, 1)$ of type (2, 2, 0) which satisfies the following conditions:

(i) The reduct $(H, \to, 1)$ is a Hilbert algebra.

(ii) These identities are verified:

(1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

(2) $x \wedge x = x$;

(3) $x \wedge (x \to y) = x \wedge y$;

(4) $(x \to (y \wedge z)) \to ((x \to z) \wedge (x \to y)) = 1$.

Example 2.6. Let $(H, \wedge, 1)$ be a meet- semilattice with element 1. Then $(H, \wedge, \to, 1)$ which

$$x \to y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$$

is a Hilbert algebra with infimum.

It is proved that the order determined by the structure of the meet-semilattice coincides with the order determined by the structure of a Hilbert algebra [9]. Thus $x \le y$ if and only if $x = x \wedge y$ if and only if $x \to y = 1$.

We say that a Hilbert algebra with infimum has the property (P) if for every $x, y \in H$,

$$(P): \; x \to (y \to (x \wedge y)) = 1$$

Theorem 2.7. *[4] Let H be a Hilbert algebra with infimum. The following assertions are equivalent:*

(i) H has the property (P);

(ii) For every $x, y, z \in H$, $x \to (y \to z) = (x \wedge y) \to z$;

(iii) For every $x, y, z \in H$, $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$.

We note that the Hilbert algebras with infimum satisfying the property (P) are *Brouwerian semilattices* [15], also called *Implicative semilattices* [18], or also called *Hertz algebras* [20].

We recall that an Implicative semilattice [6] is an algebra $(A, \wedge, \to, 1)$ in which $(A, \wedge)$ is a semilattice and for all $x, y, z \in A$,

$$z \leq x \to y \text{ if and only if } z \wedge x \leq y.$$

If H is a Hilbert algebra with infimum satisfying the property (P), then

$$x \to (y \to z) = (x \wedge y) \to z.$$

So, $x \to (y \to z) = 1$ iff $(x \wedge y) \to z = 1$, and as $a \leq b$ iff $a \to b = 1$, for all $a, b \in H$, we get that $x \leq y \to z$ iff $x \to (y \to z) = 1$ iff $(x \wedge y) \to z = 1$ iff $(x \wedge y) \leq z$. Thus, H is an implicative semilattices.

Definition 2.8. [7] An algebra $(H, \to, \vee, 1)$ of type (2, 2, 0) is a Hilbert algebra with supremum or sH-Hilbert algebra if

(i) $(H, \to, 1)$ is a Hilbert algebra;

(ii) $(H, \vee, 1)$ is a join-semilattice with last element 1.

(iii) For all $a, b \in H, a \to b = 1$ if and only if $a \vee b = b$.

Example 2.9. [7]

(i) In every join-semilattice $(H, \vee, 1)$ with last element 1 it is possible to define a structure of Hilbert algebra with supremum considering the implication $\to$ defined by the order, i.e., $a \to b = b$ if $a \nleq b$, and $a \to b = 1$ if $a \leq b$.

(ii) The Boolean lattice with two atoms $B_2 = \{0, a, b, 1\}$ with the implication $\to$ defined by the order is a Hilbert algebra where the supremum exists for any pair of elements but it is not a Heyting algebra.(An atom of a Boolean algebra is an element x such that there exist exactly two elements y satisfying $y \leq x$, namely x and 0.)

Remark 2.10. [2] If H is a Hilbert algebra without 0, then by adding a new element $0 \notin H$ and define in $H' = H \cup \{0\}$ the implication as in table:

$\rightarrow$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

(where $x \in H$), then $(H', \rightarrow, 0, 1)$ becomes a bounded Hilbert algebra.

Definition 2.11. [1, 2, 8] A subset D of Hilbert algebra H is called a deductive system (or implication filter or simply filter) of H if

(1) $1 \in D$,

(2) If $x \in D$ and $x \rightarrow y \in D$, then $y \in D$,

for all $x, y \in H$.

We note that, in some papers the deductive system is called implicative filter too. We use implicative filter in this paper.

Theorem 2.12. *[8] Let D be a deductive system of Hilbert algebra. If $x \leq y$ and $x \in D$, then $y \in D$.*

We denote $Ds(H)$= $\{D$: D is a deductive system of $H\}$. If H is bounded, then a deductive system D is proper if and only if $0 \notin D$. From now $(H, \rightarrow, 0, 1)$ or simply H is a Hilbert algebra.

Theorem 2.13. *[12] Let D be a nonempty subset of H. Then D is an implicative filter of H if and only if $x \leq y \rightarrow z$ implies $z \in D$, for $x, y \in D$.*

Let F be an implicative filter of a bounded Hilbert algebra H. Define: $x \equiv_F y$ if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$. Then $\equiv_F$ is a congruence relation on H. The set of all congruence classes is denoted by H/F, i.e, $H/F := \{[x] : x \in H\}$, where $[x] = \{y \in H | x \equiv_F y\}$. Define $\rightarrow$ on H/F as follows: $[x] \rightarrow [y] = [x \rightarrow y]$, and $1 = 1/F = F$.

Therefore $(H/F, \rightarrow, [1], [0])$ is a bounded Hilbert algebra with respect to F and the order relation on H/F is given by $[x] \leq [y]$ if and only if $x \rightarrow y \in F$. Clearly, $[x] = [1]$ if and only if $x \in F$.

Theorem 2.14. *[4] For a bounded Hilbert algebra H, the following assertions are equivalent:*

(i) $x^{**} = x$, *for every* $x \in H$*;*

(ii) H *is a Boolean algebra relative to natural ordering, where* $x \wedge y = (x \to y^*)^*$, $x \vee y = x^* \to y$.

Corollary 2.15. *Let* $(H, \to, 0, 1)$ *be a bounded Hilbert algebra, which* $x^{**} = x$*, for every* $x \in H$*. Then*

(i) $(H, \vee, \wedge, \to, 0, 1)$ *is a Heyting algebra;*

(ii) $(H, \vee, \wedge, \to, *, 0, 1)$ *is a Boolean algebra.*

Corollary 2.16. *[4] A bounded Hilbert algebra* H *is a Boolean algebra (relative to natural ordering) if and only if for every* $x, y \in H$ *we have* $(x \to y) \to x = x$.

Corollary 2.17. *[4] For a bounded Hilbert algebra* H*, the following assertions are equivalent:*

(i) H *is Boolean algebra (relative to natural ordering);*

(ii) $(x \to y) \to y = (y \to x) \to x$*;*

(iii) $x^* \to y = y^* \to x$*;*

(iv) $(x \to y) \to y = x \vee y$*;*

(v) $x^* \to y = x \vee y$.

Definition 2.18. [1] An implicative filter M of a Hilbert algebra H is called maximal if it is not properly contained in any other proper implicative filter of H.

Theorem 2.19. *[1] An implicative filter* M *of a bounded Hilbert algebra* H *is a maximal if and only if for all* $x \in H$*, if* $x \notin M$ *then* $x^* \in M$.

3 Implicative filters in Hilbert algebras

In this section we give simple characterizations of implicative filters defined in the previous section.

Theorem 3.1. *Let* $D \subseteq H$*,* $1 \in D$*. Then the following conditions are equivalent, for every* $x, y, z \in H$*:*

(i) D *is an implicative filter;*

(ii) *If* $x \to (y \to z) \in D$, $x \to y \in D$, *then* $x \to z \in D$*;*

(iii) *If* $x \to y$, $y \to z \in D$, *then* $x \to z \in D$*;*

(iv) $H_a = \{x \in H : a \to x \in D\}$ *is an implicative filter, for any* $a \in H$*.*

The following theorem give a characterization for maximal implicative filter in Hilbert algebras.

Theorem 3.2. *[16] Let* D *be an implicative filter of* H*. Then the following conditions are equivalent:*

(i) D *is maximal;*

(ii) D *is proper (i.e.* $D \neq H$*), and if* $x, y \notin D$*, then* $x \to y \in D$*.*

Therefore, in a bounded Implicative semilattice, we have:

Theorem 3.3. *If* D *is an implicative filter of a bounded Implicative semilattice then the following conditions are equivalent:*

(i) $y \to x \in D$ *implies* $((x \to y) \to y) \to x \in D$*, for all* $x, y \in H$*;*

(ii) $x^{**} \to x \in D$*, for every* $x \in H$*;*

(iii) *If* $x \to u \in D$ *and* $y \to u \in D$*, then* $((x \to y) \to y) \to u \in D$*, for every* $x, y, u \in H$*.*

4 Positive implicative filters in Hilbert algebras

In this section, the notion of a positive implicative filter in Hilbert algebras is introduced and its properties are studied.

Definition 4.1. A non- empty subset F of H is called a positive implicative filter if it satisfies:

(i) $1 \in F$,

(ii) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$,

for every $x, y, z \in H$.

Example 4.2. (*i*) Define

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

Then $H = ([0,1], \to, 0, 1)$ is a Hilbert algebra. $F = [\frac{1}{2}, 1]$ is an implicative filter but is not a positive implicative filter, since $\frac{2}{3} \to ((\frac{1}{3} \to \frac{1}{4}) \to \frac{1}{3}) = 1 \in F$ and $\frac{2}{3} \in F$ but $\frac{1}{3} \notin F$.

(*ii*) Let $H = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, and a, b are incompatible. With the following operation:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

H is a Hilbert algebra and {1, c} is a positive implicative filter.

(*iii*) Let $H = \{1, a, b, c\}$. Define $\to$ by

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then H is a Hilbert algebra and by routine calculations we can see that $\{1\}$ is a positive implicative filter.

Proposition 4.3. *Let F be a positive implicative filter. Then, for all $x, y \in H$,*

(*i*) $(x \to y) \to x \in F$, *implies* $x \in F$.

(*ii*) $(x \to y) \to y \in F$ *implies* $(y \to x) \to x \in F$.

Proposition 4.4. *Let F be an implicative filter of H and $(x \to y) \to x \in F$, implies $x \in F$, for all $x, y \in H$. Then F is a positive implicative filter of H.*

Proposition 4.5. *Let F be an implicative filter of H and $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for all $x, y \in H$. Then F is a positive implicative filter of H.*

Theorem 4.6. *The following conditions are equivalent:*

(*i*) *$\{1\}$ is a positive implicative filter of H;*

(*ii*) *Every implicative filter of H is a positive implicative filter;*

(*iii*) *$H(a) = \{x \in H : x \geq a\}$ is a positive implicative filter, for every $a \in H$;*

(*iv*) *$(x \to y) \to x = x$, for all $x, y \in H$.*

5 Fantastic filters in Hilbert algebras

In this section, the notion of fantastic filters in Hilbert algebra is defined. We give relationships between fantastic filters and other types of filters in H.

Definition 5.1. A non- empty subset F of H is called a fantastic filter if it satisfies:

(*i*) $1 \in F$,

(*ii*) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$,

for every $x, y, z \in H$.

Theorem 5.2. *Any fantastic filter of H is an implicative filter of H.*

In the following example, we show that the converse of above theorem is not true.

Example 5.3. (*i*) Let $H = \{1, a, b, c, d\}$. In which $\to$ is defined by

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	1	1	1	d
d	1	1	b	c	1

$(H, \to, 1)$ is a Hilbert algebra, $F = \{1\}$ is an implicative filter, but is not fantastic filter, since $1 \to (c \to b) \in F, 1 \in F$. But $((b \to c) \to c) \to b = b \notin F$.

(ii) Let $H = \{1, a, b, c\}$. The operation $\to$ defined by:

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then H is a Hilbert algebra and $\{1, c\}$ is a fantastic filter.

Now, we give an equivalent condition for fantastic filter.

Theorem 5.4. *Let F be an implicative filter of H. Then F is fantastic filter if and only if $x \to y \in F$ implies $((y \to x) \to x) \to y \in F$, for all x, $y \in H$.*

In the following theorem we describe the relationship between positive implicative filters and fantastic filters.

Theorem 5.5. *If F is a positive implicative filter of H, then F is a fantastic filter of H.*

The converse of Theorem 5.7, is correct in bounded Hilbert algebra (see Proposition 6.7).

Theorem 5.6. *$\{1\}$ is a fantastic filter if and only if $((x \to y) \to y) \to x = y \to x$, for all x, $y \in H$.*

6 Boolean filters in Hilbert algebra

In this section, H is a bounded Hilbert algebra with supremum, unless otherwise is stated.

Definition 6.1. An implicative filter F of H is called a Boolean filter of:

(1) first kind $(BF1)$ if $x^{**} \to x \in F$, for all $x \in H$.

(2) second kind ($BF2$) if $x \vee x^* \in F$, for all $x \in H$.

Example 6.2. *(i)* Let $H = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, but a, b are incompatible. We define

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$(H, \rightarrow, 0, 1)$ is a bounded Hilbert algebra with supremum. The implicative filters are $\{1\}, \{c, 1\}, \{a, c, 1\}, \{b, c, 1\}$ and $\{0, a, b, c, 1\}$. The implicative filters $\{c, 1\}$, $\{a, c, 1\}, \{b, c, 1\}, \{0, a, b, c, 1\}$ are $BF1$ and $BF2$, $\{1\}$ is a $BF1$ but is not a $BF2$.

(ii) Let $H = \{0, a, b, 1\}$. $\rightarrow$ defined by:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then H is a bounded Hilbert algebra with supremum, $F = \{1\}$ is a $BF1$, but is not a $BF2$, because $a \vee b = 1$ but $a, b \notin F$. The filters $\{a, 1\}, \{b, 1\}, \{0, a, b, 1\}$ are $BF1$ and $BF2$, but their intersection $\{1\}$ is not a $BF2$.

(iii) Let $H = \{0, a, 1\}$, where $0 < a < 1$. The operation $\rightarrow$ defined by:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Notice that $\{1\}$ is not $BF1$ and $BF2$, because $a^{**} \rightarrow a = a \notin \{1\}$ and $a \vee a^* = a \notin \{1\}$.

(*iv*) Let $H = \{0, a, b, c, 1\}$. We define

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

$(H, \to, 0, 1)$ is a bounded Hilbert algebra with supremum. The implicative filter $\{1\}$ is not none of $BF1$ and $BF2$.

Proposition 6.3. *In a bounded Hilbert algebra with supremum H every Boolean filter of the second kind is a positive implicative filter.*

In the following example, we show that the converse of Proposition 6.3 is not true.

Example 6.4. In Example 6.2,(*i*), $\{1\}$ is a positive implicative filter, but is not a Boolean filter of the second kind.

Proposition 6.5. *If F is a fantastic filter in a bounded Hilbert algebra H, then F is a Boolean filter of the first kind.*

Example 6.6. In Example 5.3(ii), $\{c, 1\}$ is a fantastic filter but is not a Boolean filter of the first kind since H is not bounded.

Proposition 6.7. *In a bounded Hilbert algebra H, positive implicative filter and fantastic filter are equivalent.*

Open problem. Does the Proposition 6.7 hold in every Hilbert algebra?

Theorem 6.8. *In a bounded Hilbert algebra H, positive implicative filter, fantastic filter and Boolean filter of the first kind are equivalent.*

Theorem 6.9. *Let H be a bounded Hilbert algebra. The following conditions are equivalent:*

(*i*) $\{1\}$ *is a Boolean filter of the first kind,*

(*ii*) *Every implicative filter of H is a Boolean filter of the first kind,*

(iii) $H(a) = \{x \in H : x \geq a\}$ *is a Boolean filter of the first kind, for every* $a \in H$,

(iv) $(x \to y) \to x = x$ *for all* $x, y \in H$,

(v) $x^{**} = x$,

(vi) H *is a Boolean algebra.*

Theorem 6.10. *Let H be a bounded Hilbert algebra with supremum. The following conditions are equivalent:*

(i) $\{1\}$ *is a Boolean filter of the second kind,*

(ii) *Every implicative filter of H is a Boolean filter of the second kind,*

(iii) $H(a) = \{x \in H : x \geq a\}$ *is a Boolean filter of the second kind, for every* $a \in H$.

Theorem 6.11. *Let H be a bounded Hilbert algebra. Then $\{1\}$ is a Boolean filter of the first kind of H if and only if $((x \to y) \to y) \to x = y \to x$, for all $x, y \in H$.*

Theorem 6.12. *Let F be a maximal filter of a bounded Hilbert algebra H. Then F is a Boolean filter of the first kind.*

Theorem 6.13. *Let F be a maximal filter of a bounded Hilbert algebra with supremum H. Then F is a Boolean filter of the second kind.*

7 Prime filters in Hilbert algebra

In this section H is a Hilbert algebra with supremum, unless otherwise is stated.

Definition 7.1. An implicative filter F of H is called a prime filter of:

(1) first kind ($PF1$), if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in H$.

(2) second kind ($PF2$), if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in H$.

(3) third kind ($PF3$), if $(x \to y) \vee (y \to x) \in F$, for all $x, y \in H$.

Example 7.2. (i) In Example 6.2 (i), the implicative filters are $\{1\}$, $\{c, 1\}, \{a, c, 1\}, \{b, c, 1\}$ and $\{o, a, b, c, 1\}$. Remark that $\{1\}$ is $PF1$, but is not a $PF2$, since $a \to b = b \notin \{1\}$, $b \to a = a \notin \{1\}$. Also note that $\{1\}$ is not $PF3$, since $(a \to b) \vee (b \to a) = b \vee a = c$. The implicative filter $\{c, 1\}$ is $PF3$, but is not $PF2$ and $PF1$, since $a \to b$ and $b \to a \notin \{c, 1\}$ also, $a \vee b = c \in \{c, 1\}$ but $a, b \notin \{c, 1\}$. The implicative filters $\{a, c, 1\}, \{b, c, 1\}, \{o, a, b, c, 1\}$ are $PF1$, $PF2$, $PF3$, but their intersection $\{c, 1\}$ are not $PF2$, $PF1$.

(ii) In Example 6.2 (ii), $F = \{1\}$ is a Boolean filter of the first kind, but is not $PF1$, because $a \vee b = 1$ but $a, b \notin F$. The filters $\{a, 1\}, \{b, 1\}, \{o, a, b, 1\}$ are $PF1$, $PF2$, $PF3$, $BF1$ and $BF2$, but their intersection $\{1\}$ is not $PF1$, $PF2$ and $BF2$.

(iii) In Example 6.2 (iii), $F = \{1\}$ is a prime filter of the first kind, but is not $BF1$, because $a \vee a^* = a \notin \{1\}$.

(iv) In Example 6.2 (iv), $(H, \to, 0, 1)$ is a bounded Hilbert algebra with supremum. The implicative filter $\{1\}$ is $PF3$, but is not $PF1$, $PF2$, $BF1$ and $BF2$.

In the following theorem we state the relationship between all kinds of prime filters.

Theorem 7.3. *For an implicative filter P of H, consider the following assertions:*

(1) P is a prime filter of the first kind;

(2) P is a prime filter of the second kind;

(3) P is a prime filter of the third kind.

Then

(i) (2) $\Rightarrow$ (1) *but* (1) $\not\Rightarrow$ (2)*;*

(ii) (2) $\Rightarrow$ (3) *but* (3) $\not\Rightarrow$ (2)*;*

(iii) (1) + (3) $\Leftrightarrow$ (2)*.*

Proposition 7.4. *If H is prelinear, then every prime filter of the first kind is a prime filter of the second kind.*

Remark 7.5. Let F be a prime filter of the first kind of a bounded Hilbert algebra with supremum H. If $(x \to y) \vee (y \to x) = 1$, then $x \to y \in F$ or $y \to x \in F$, for all $x, y \in H$.

Theorem 7.6. *Every maximal filter of H is a prime filter of the second kind.*

Corollary 7.7. *Every maximal filter of H is a prime filter of the first kind.*

In the following example, we show that the converse of Theorem 6.12 and Corollary 7.7 is not true. Also we show that extension property for prime filters is not hold.

Example 7.8. In Example 7.2(i), $H = \{0, a, b, c, 1\}$ is a Hilbert algebra, $\{1\}$ is a prime filter of the first kind. Since $c^{**} \to c = 1 \to c = c \notin \{1\}$, so $\{1\}$ isn't a Boolean filter of the first kind, thus is not a maximal filter. Also $\{c, 1\}$ is Boolean filter of the first kind but it is not a prime filter of the first kind since $a \vee b = c \in \{c, 1\}$ but $a, b \notin \{c, 1\}$. Also $\{c, 1\}$ is not a maximal filter, since $a \to b = b \notin \{c, 1\}$.

Theorem 7.9. *Let $(H \to, 1)$ be a Tarski algebra where H is bounded. An implicative filter F is maximal filter if and only if F is prime and Boolean filter of the first kind of H.*

Theorem 7.10. *Let F be an implicative filter of bounded Hilbert algebra H. F is maximal filter if and only if F is a prime filter of the first kind and a Boolean filter of the second kind.*

Theorem 7.11. *Every Boolean filter of the second kind is a prime filter of the third kind.*

Proof. For any two elements x and y we have $x \leq y \to x$ and $x^* \leq x \to y$, it follows that every Boolean filter of the second kind of a Hilbert algebra is also a prime filter of the third kind.

□

Theorem 7.12. *For a proper implicative filter P of H consider the following assertions:*

(1) P is a prime filter of the first kind;

(2) If $a, b \in H$, and $a \vee b = 1$, then $a \in P$ or $b \in P$;

(3) For all $a, b \in H, a \to b \in P$ or $b \to a \in P$;

(4) H/P is a chain.

Then

(i) $(1) \Rightarrow (2)$ *but* $(2) \nRightarrow (1)$, $(2) \nRightarrow (3)$, $(2) \nRightarrow (4)$*;*

(ii) $(3) \Rightarrow (1)$ *but* $(1) \nRightarrow (3)$*;*

(iii) $(4) \Rightarrow (1)$ *but* $(1) \nRightarrow (4)$*;*

Remark 7.13. In Theorem 7.12, if $(a \to b) \vee (b \to a) = 1$, then $(1) \Leftrightarrow (3) \Leftrightarrow (4)$.

Theorem 7.14. *Let P be a proper implicative filter of H. Then P is a prime filter of the second kind if and only if H/P is a chain.*

8 Implication filter in Hilbert algebra

From now $(H, \to, 0, 1)$ or simply H is a Hilbert algebra.

Definition 8.1. An implicative filter F of H is called implication filter, if satisfy:

$((x \to y) \to x) \to x \in F$, for all $x, y \in H$ such that $y \neq 0$.

(If $y = 0$, then $((x \to 0) \to x) \to x \in F$, so $(x^* \to x) \to x = x^{**} \to x \in F$. Thus F is a Boolean filter of H.)

Example 8.2. *(i)* Let $H = \{0, a, b, c, 1\}$. We define the following operations:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

then H is a Hilbert algebra, $F = \{a, 1\}$ is an implication filter of H, but it is not a Boolean filter, since $b^{**} \to b = b \notin F$.

(ii) Let $H = \{0, a, b, c, d, e, f, g, 1\}$, with $0 < a < b < e < 1, 0 < a < d < e < 1, 0 < a < d < g < 1, 0 < c < d < e < 1, 0 < c < d < g < 1, 0 < c < f < g < 1$ but elements $\{a, c\}, \{b, d\}, \{d, f\}, \{e, g\}$ and $\{b, f\}$ are pairwise incompatable. Define

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

Then $(H, \to, 0, 1)$ is a Hilbert algebra and $F = \{1\}$ is an implication filter.

Theorem 8.3. *In H the following conditions are equivalent:*
(i) $\{1\}$ is an implication filter of H;
(ii) Every implicative filter of H is an implication filter;
(iii) $((x \to y) \to x) \to x = 1$, for all $x, y \in H$ such that $y \neq 0$;
(iv) $(x \to y) \to x = x$, for all $x, y \in H$ such that $y \neq 0$.

Theorem 8.4. *If F is an implication filter of H, then $Rad(F)$ is an implication filter of H.*

In the following example we show that the converse of above theorem is not correct.

Example 8.5. Let $H = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, but a, b are incomparable. We define

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(H, \to, 0, 1)$ is a bounded Hilbert algebra. The implicative filter $F = \{1\}$ is not an implication filter, since $((c \to a) \to c) \to c = (a \to c) \to c = 1 \to c = c \notin F$. But, $Rad(F)$ is an implication filter of H.

9 Implication Hilbert algebra

Definition 9.1. A bounded Hilbert algebra H is called implication Hilbert algebra if it satisfies in the following condition:
(I) $(x \to y) \to x = x$, for all $x, y \in H$ such that $y \neq 0$.

(If $y = 0$, then $(x \to 0) \to x = x^{**} = x$. Thus H is a Boolean algebra.)

Example 9.2. (i) In Example 8.2 (i), $H = \{0, a, b, c, 1\}$ is an implication Hilbert algebra.

(ii) In Example 8.2 (ii), $H = \{0, a, b, c, d, e, f, g, 1\}$ is an implication Hilbert algebra.

(iii) Let $H = \{0, a, b, c, 1\}$, with $0 < a, b < c < 1$, but a, b are incompatable. We define

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(H, \to, 0, 1)$ is a bounded Hilbert algebra, but it is not implication Hilbert algebra, because $(c \to a) \to c = 1 \neq c$.

Proposition 9.3. *Every linear Hilbert algebra H is not an implication Hilbert algebra.*

Example 9.4. Let $H = \{0, a, b, 1\}$. Define on H the following operation:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then H is an implication Hilbert algebra, but it is not a linear Hilbert algebra.

Theorem 9.5. *F is an implication filter of H if and only if H/F is an implication Hilbert algebra.*

Corollary 9.6. *If F is a Boolean filter of the first (second) kind of H, then H/F is an implication Hilbert algebra.*

Corollary 9.7. *The following conditions are equivalent:*

(i) *Every implicative filter of H is an implication filter of H;*

(ii) *$\{1\}$ is an implication filter of H;*

(iii) *H is an implication Hilbert algebra.*

Proposition 9.8. *If F is a Boolean filter of the first (second) kind of H, then F is an implication filter of H.*

In the following we show that the converse of above proposition is not correct.

Example 9.9. In Example 9.2(i), H is an implication Hilbert algebra, $F = \{a, 1\}$ is an implication filter, but it is not a Boolean filter of the first kind, since $b^{**} \rightarrow b = b \notin F$.

Corollary 9.10. *If F is a maximal filter (positive implicative, fantastic) of H, then F is an implication filter of H.*

In [21] An implication algebra is a Hilbert algebra $(H, \rightarrow, 1)$ such that $(x \rightarrow y) \rightarrow x = x$, for all $x, y \in H$.

In [21] Hilbert algebra is called positive implication algebra.

Theorem 9.11. *In a bounded Hilbert algebra H, an implication algebra and Boolean algebra are equivalent.*

In the following example, we show that bounded condition in above theorem is essential.

Example 9.12. Let $H = \{a, b, c, d, 1\}$, we define

$\rightarrow$	a	b	c	d	1
a	1	1	1	d	1
b	c	1	c	d	1
c	b	b	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Then $(H, \rightarrow, 1)$ is an implication algebra, since H is a Hilbert algebra such that $(x \rightarrow y) \rightarrow x = x$, for all $x, y \in H$. But it is not a Boolean algebra, since H is not bounded.

In([10], [20]) A Hertz algebra is an algebra $(A, \rightarrow, \wedge, 1)$ of type (2, 2, 0) which satisfies the following axioms: (i) $x \rightarrow x = 1$; (ii) $(x \rightarrow y) \wedge y = y$; (iii) $x \wedge (x \rightarrow y) = x \wedge y$; (iv) $x \rightarrow (y \wedge z) = (x \rightarrow z) \wedge (x \rightarrow b)$. In [9] it is proved for a Hilbert algebra H, H is a Hertz algebra if and only if H is a Hilbert algebra with infimum which verifies property (P).

10 Conclusion and future research

In the following first diagram (figure 1) we summarize the results of this paper and the previous results in this field and we give the relationships between all types of filters in a Hilbert algebra. Also, in the following second diagram (figure 2) we give the relationships between Hilbert algebra and other algebraic structures. By $A \longrightarrow B$ ($A \xrightarrow{a} B$ or $A \xrightarrow[a]{} B$), we means that A implies B (respectively, A implies B with the condition "a").

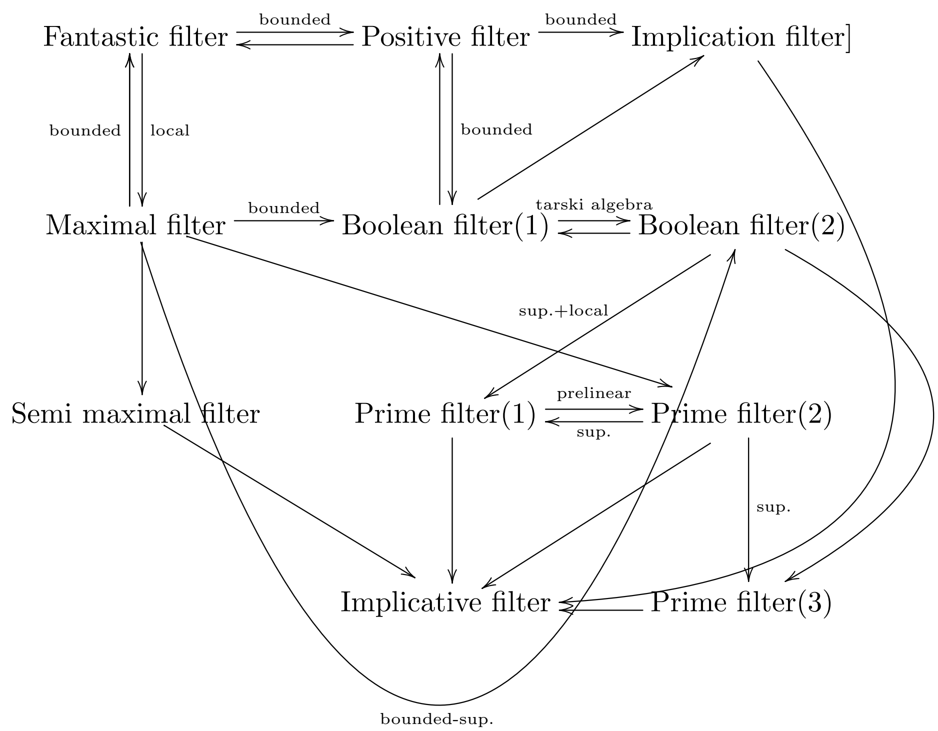

figure 1: The relationships between all types of filters in a Hilbert algebra.

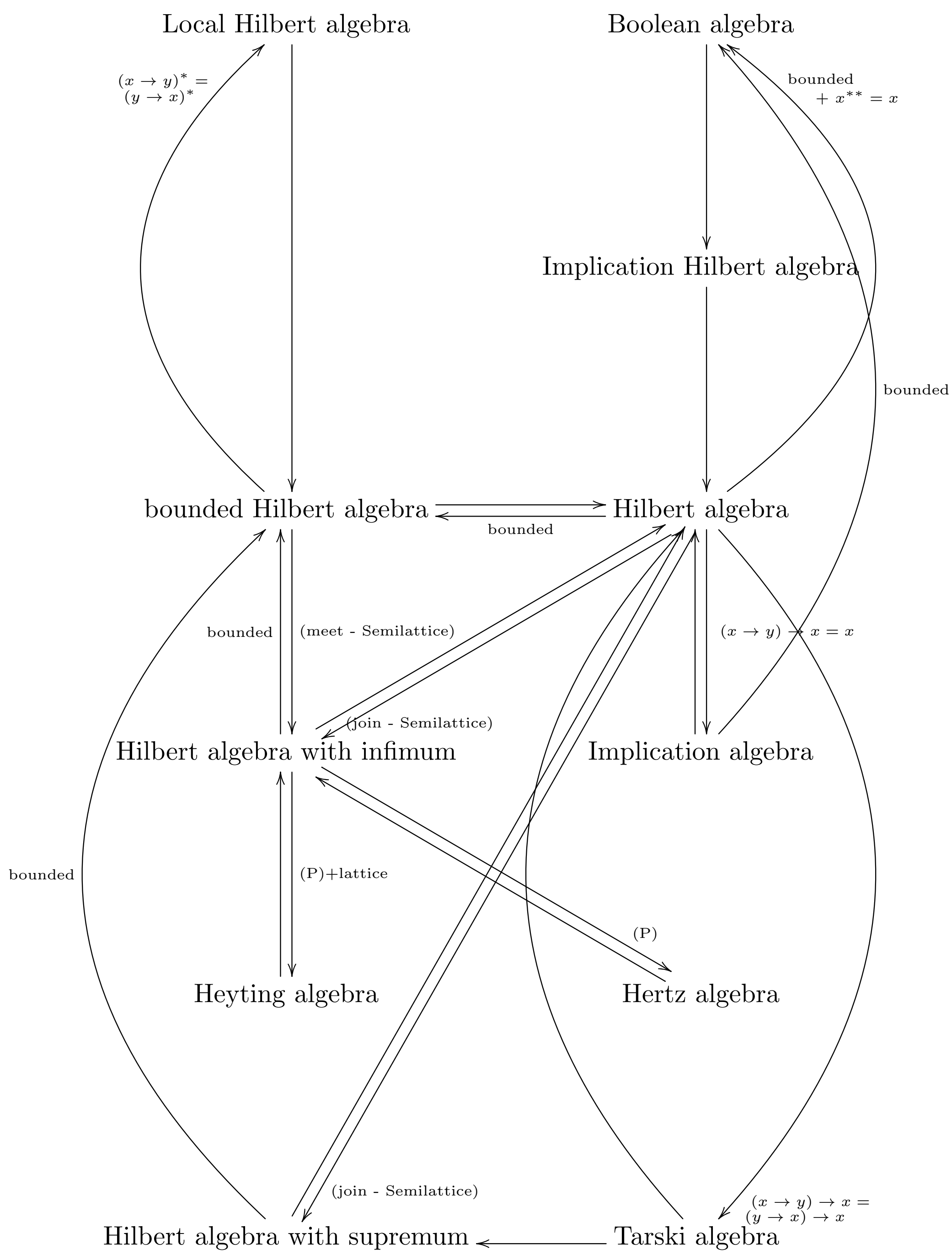
Local Hilbert algebra
Boolean algebra
$(x \to y)^* =$
$(y \to x)^*$
bounded
$+ x^{**} = x$
Implication Hilbert algebra
bounded
bounded Hilbert algebra
Hilbert algebra
bounded
bounded
(meet - Semilattice)
$(x \to y) \to x = x$
(join - Semilattice)
Hilbert algebra with infimum
Implication algebra
bounded
(P)+lattice
(P)
Heyting algebra
Hertz algebra
(join - Semilattice)
$(x \to y) \to x =$
$(y \to x) \to x$
Hilbert algebra with supremum
Tarski algebra

figure 2: The relationships between Hilbert algebra and other algebraic structures.

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