The homomorphism lattice induced by a finite algebra

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The homomorphism order on a category

- Let $\mathcal{C}$ be a category.
- Define a quasi-order $\to$ on the objects of $\mathcal{C}$ by $\mathbf{X} \to \mathbf{Y}$ if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$.
- Now set
$$\mathbf{X} \equiv \mathbf{Y} \iff \mathbf{X} \to \mathbf{Y} \text{ and } \mathbf{Y} \to \mathbf{X}.$$
  Then $\to$ induces an order on the class $\mathcal{C}/\equiv$.

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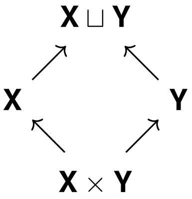
Then $\to$ induces an order on the class $\mathcal{C}/{\equiv}$.
- To ensure that $\mathcal{C}/{\equiv}$ is a set we take $\mathcal{C}$ to be a class of finite structures with all homomorphisms between them.
- Then we can define the ordered set
$$\mathbf{P}_{\mathcal{C}} := \langle \mathcal{C}/{\equiv}; \to \rangle,$$
which we refer to as the homomorphism order on $\mathcal{C}$.

Categories with product and coproduct

- $\mathbf{P}_{\mathcal{C}} := \langle \mathcal{C}/{\equiv}; \to \rangle$, where $\mathbf{X} \equiv \mathbf{Y} \iff \mathbf{X} \to \mathbf{Y}$ and $\mathbf{Y} \to \mathbf{X}$.
- If the category $\mathcal{C}$ has pairwise products and coproducts, then $\mathbf{P}_{\mathcal{C}}$ is a lattice:
  - the meet of $\mathbf{X}/{\equiv}$ and $\mathbf{Y}/{\equiv}$ is $(\mathbf{X} \times \mathbf{Y})/{\equiv}$,
  - the join of $\mathbf{X}/{\equiv}$ and $\mathbf{Y}/{\equiv}$ is $(\mathbf{X} \sqcup \mathbf{Y})/{\equiv}$.



Some examples

- The case where $\mathcal{C}$ is the category of finite graphs or finite digraphs has been studied extensively:
  - in both cases, $\mathbf{P}_{\mathcal{C}} = \langle \mathcal{C}/{\equiv}; \to \rangle$ is a distributive lattice (in fact, a Heyting algebra) into which every countable ordered set can be embedded.

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  - in both cases, $\mathbf{P}_{\mathcal{C}} = \langle \mathcal{C}/{\equiv}; \to \rangle$ is a distributive lattice (in fact, a Heyting algebra) into which every countable ordered set can be embedded.
- We are interested in the case where $\mathcal{C}$ is a category of finite algebras.
- For many natural categories of finite algebras, the ordered set $\mathbf{P}_{\mathcal{C}}$ is trivial as there are homomorphisms between every pair of objects in $\mathcal{C}$. For example:
  - groups, vector spaces over a finite field, rings, lattices, semigroups.

Locally finite varieties

- Let $\mathcal{A}$ be a variety of algebras and let $\mathcal{A}_{\text{fin}}$ denote the class of finite members of $\mathcal{A}$.
- If $\mathcal{A}_{\text{fin}}$ is closed under forming finite coproducts, then $\mathbf{P}_{\mathcal{A}_{\text{fin}}}$ is a lattice.
- Hence, it is natural to look at locally finite varieties.

Lemma

Let $\mathcal{A}$ be a locally finite variety. Then the homomorphism order $\mathbf{P}_{\mathcal{A}_{\text{fin}}} = \langle \mathcal{A}_{\text{fin}}/{\equiv}; \to \rangle$ is a countable bounded lattice.

Note

- The top of $\mathbf{P}_{\mathcal{A}_{\text{fin}}}$ is the equivalence class consisting of algebras in $\mathcal{A}_{\text{fin}}$ that have a trivial subalgebra.
- The bottom of $\mathbf{P}_{\mathcal{A}_{\text{fin}}}$ is the equivalence class that contains all finitely generated free algebras in $\mathcal{A}$.

The homomorphism lattice induced by a finite algebra

Fact

Let $\mathbf{A}$ be a finite algebra. Then the variety $\mathrm{Var}(\mathbf{A})$ generated by $\mathbf{A}$ is locally finite.

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Let **A** be a finite algebra. Then the variety Var(**A**) generated by **A** is locally finite.

Notation
Given a finite algebra **A**, we define the lattice

$$\mathbf{L}_{\mathbf{A}} := \langle \mathsf{Var}(\mathbf{A})_{\text{fin}}/\!\equiv; \to \rangle$$

and refer to it as the homomorphism lattice induced by **A**.

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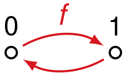
Question

- Which lattices arise as $\mathbf{L}_{\mathbf{A}}$ for some finite algebra **A**?

Warm-up exercises

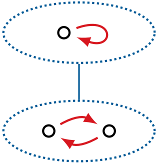
Example 1. Representing finite chains

- The 2-element chain:



$\mathbf{A} = \langle \{0, 1\}; f \rangle$

$\mathbf{L_A} \cong \mathbf{2}$



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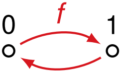

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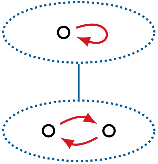
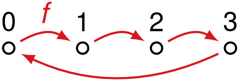


- The 3-element chain:



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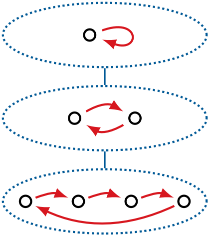

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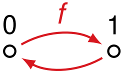

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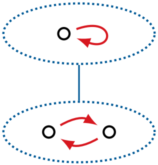

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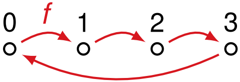

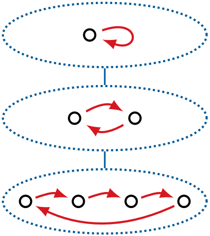

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$\mathbf{L_A} \cong \mathbf{3}$

- In general, if $\mathbf{A} = \langle \{0, 1, \dots, 2^{n-1} - 1\}; f \rangle$, where f is successor modulo 2^{n-1}, then $\mathbf{L_A} \cong \mathbf{n}$.

An interlude on cores

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- If we let $\mathcal{K}$ consist of one copy (up to isomorphism) of each core in $\mathrm{Var}(\mathbf{A})_{\mathrm{fin}}$, then $\mathcal{K}$ is a transversal of the equivalence classes of $\mathbf{L}_{\mathbf{A}}$.
- Moreover, the cores are precisely the algebras that are minimal-sized within their equivalence classes.
- So the lattice $\mathbf{L}_{\mathbf{A}}$ is finite if and only if there is a finite bound on the sizes of the cores in $\mathrm{Var}(\mathbf{A})_{\mathrm{fin}}$.

Warm-up exercises

Example 2. The homomorphism lattice induced by a finite monounary algebra

Questions

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Lemma

Let $\mathbf{A} = \langle A; f \rangle$ be a finite monounary algebra. Then, for some ℓ and some $k_1, \ldots, k_\ell$,

$$\mathbf{L_A} \cong (\mathbf{k}_1 \sqcup \cdots \sqcup \mathbf{k}_\ell) \oplus \mathbf{1},$$

where the coproduct is in the variety $\mathfrak{D}$ of distributive lattices. (*Let n be the least common multiple of the sizes of the cycles of $\mathbf{A}$. Then $p_1^{k_1} \cdots p_\ell^{k_\ell}$ is the prime decomposition of n.*)

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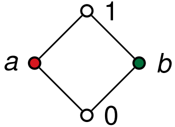
- So the answer to Question 2 is 'No'.

Warm-up exercises

Example 3. Representing the lattice $\mathbf{2}^2$

$$\mathbf{A} = \langle \{0, a, b, 1\}; \vee, \wedge, 0, 1, a, b \rangle$$



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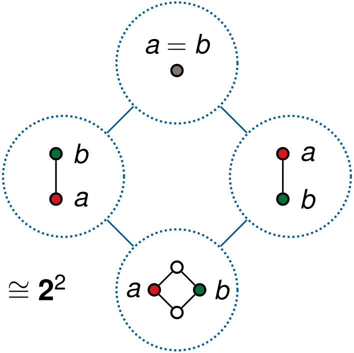
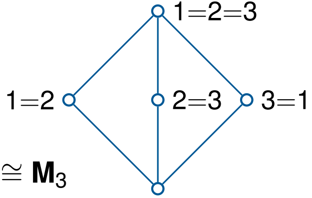


Warm-up exercises

Example 4. Representing finite partition lattices

- To represent the lattice $\mathbf{M}_3$, we can use the nullary algebra $\mathbf{A} = \langle\{1,2,3\}; 1,2,3\rangle$:



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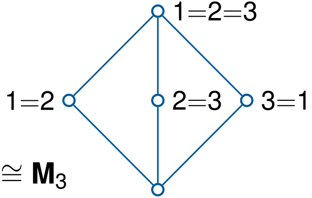

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- If $\mathbf{A} = \langle\{1, 2, \ldots, n\}; 1, 2, \ldots, n\rangle$, then $\mathbf{L_A} \cong \mathrm{Equiv}(n)$.

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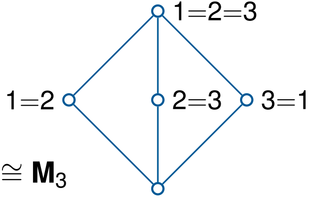

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Questions

- Can we represent each finite distributive lattice as $\mathbf{L_A}$?
- We have represented $\mathbf{M}_3$. What about $\mathbf{N}_5$?

We will answer both of these questions later in the talk.

- A variety $\mathcal{V}$ of algebras is finite-to-finite universal if the category $\mathcal{G}$ of digraphs has a finiteness-preserving full embedding into $\mathcal{V}$.

How to find complicated examples

Finite-to-finite universal varieties

- A variety $\mathcal{V}$ of algebras is finite-to-finite universal if the category $\mathcal{G}$ of digraphs has a finiteness-preserving full embedding into $\mathcal{V}$.
- This is equivalent to requiring that *every* variety of algebras has a finiteness-preserving full embedding into $\mathcal{V}$.

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- A variety $\mathcal{V}$ of algebras is finite-to-finite universal if the category $\mathcal{G}$ of digraphs has a finiteness-preserving full embedding into $\mathcal{V}$.
- This is equivalent to requiring that *every* variety of algebras has a finiteness-preserving full embedding into $\mathcal{V}$.
- If Var(**A**) is finite-to-finite universal, then
  - there is an order-embedding of $\mathbf{P}_{\mathcal{G}_{\text{fin}}}$ into $\mathbf{L}_{\mathbf{A}}$,
  - every countable ordered set order-embeds into $\mathbf{L}_{\mathbf{A}}$,
  - in particular, $\mathbf{L}_{\mathbf{B}}$ order-embeds into $\mathbf{L}_{\mathbf{A}}$, for all finite **B**.

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3. Let $\mathbf{P} = \langle\{0, 1, 2\}; \cdot\rangle$ be the semigroup shown on the right.

   Let $\mathbf{A} = \langle A; \cdot, a, b, c\rangle$, where $\langle A; \cdot\rangle$ is freely generated in Var($\mathbf{P}$) by $\{u, v, a, b, c\}$.

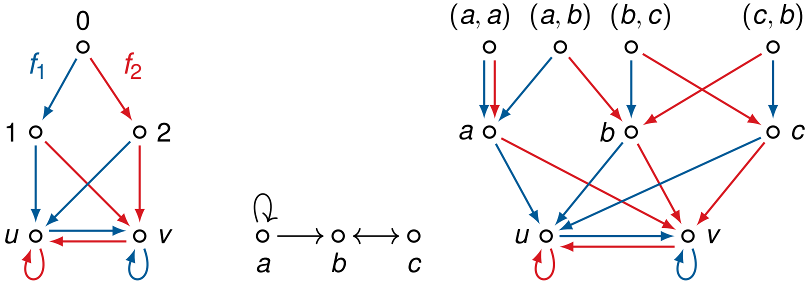
   [Uses Demlová, Koubek 2003]

·	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

Finite-to-finite universal varieties

Even a small unary algebra can generate a finite-to-finite universal variety.

4. Let $\mathbf{A} = \langle \{0, 1, 2, u, v\}; f_1, f_2 \rangle$ be the 5-element unary algebra shown below.



A **G** $\mathbf{G}^*$

The map $\mathbf{G} \mapsto \mathbf{G}^*$ embeds the category $\mathcal{G}$ of digraphs into the category Var(**A**). [Uses Hedrlín, Pultr 1966]

Distributive lattices via quasi-primal algebras

A finite algebra **A** is quasi-primal if the ternary discriminator operation τ is a term function of **A**, where

$$\tau(x, y, z) := \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

Theorem (Pixley 1970)

A finite algebra **A** *is quasi-primal if and only if every non-trivial subalgebra of* **A** *is simple and the variety* Var(**A**) *is both congruence permutable and congruence distributive.*

Theorem (Pixley 1970)

Let **A** *be a quasi-primal algebra. Then every finite algebra in* Var(**A**) *is isomorphic to a product of subalgebras of* **A**.

Distributive lattices via quasi-primal algebras

Notation

- We use $\mathrm{Sub}(\mathbf{A})$ to denote the set of all subalgebras of an algebra $\mathbf{A}$. Note that $\langle \mathrm{Sub}(\mathbf{A}); \leqslant \rangle$ is a join-semilattice.
- For an ordered set $\mathbf{P}$, we use $\mathcal{O}(\mathbf{P})$ to denote the lattice of all down-sets of $\mathbf{P}$, ordered by inclusion.

Theorem (Birkhoff 1933)

For every finite distributive lattice $\mathbf{L}$, there exists a finite ordered set $\mathbf{P}$ such that $\mathbf{L} \cong \mathcal{O}(\mathbf{P})$. Indeed,

$$\mathbf{L} \cong \mathcal{O}(\mathcal{J}(\mathbf{L})),$$

where $\mathcal{J}(\mathbf{L})$ is the ordered set of join-irreducible elements of $\mathbf{L}$.

Distributive lattices via quasi-primal algebras

Theorem

Let $\mathbf{A}$ *be a quasi-primal algebra. Define the ordered set*

$$\mathbf{Q} := \langle \mathrm{Sub}(\mathbf{A})/{\equiv}; \rightarrow \rangle$$

and let $\overline{\mathbf{Q}}$ *denote* $\mathbf{Q}$ *without its top.*

a. *If* $\mathbf{A}$ *has no trivial subalgebras, then* $\mathbf{L}_{\mathbf{A}} \cong \mathcal{O}(\mathbf{Q})$.

b. *If* $\mathbf{A}$ *has a trivial subalgebra, then* $\mathbf{L}_{\mathbf{A}} \cong \mathcal{O}(\overline{\mathbf{Q}})$.

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This result gives a strategy to show that every finite distributive lattice $\mathbf{L}$ arises as $\mathbf{L}_{\mathbf{A}}$, for some finite algebra $\mathbf{A}$:

- Let $\mathbf{P} = \mathcal{J}(\mathbf{L})$. Hence, $\mathcal{O}(\mathbf{P}) \cong \mathbf{L}$.
- Let $\mathbf{P}^{\top}$ denote $\mathbf{P}$ with a new top element $\top$ added.
- Construct a quasi-primal algebra $\mathbf{A}$, with a trivial subalgebra, such that $\langle \mathrm{Sub}(\mathbf{A})/{\equiv}; \to \rangle \cong \mathbf{P}^{\top}$. Then

$$\mathbf{L}_{\mathbf{A}} \cong \mathcal{O}(\overline{\mathbf{P}^{\top}}) = \mathcal{O}(\mathbf{P}) \cong \mathbf{L}.$$

Interlude: a motivating example

Lemma (Birkhoff, Frink 1948)

For each finite join-semilattice $\mathbf{S}$*, there exists a finite algebra* $\mathbf{A}$ *such that* $\langle \mathrm{Sub}(\mathbf{A}); \leqslant \rangle$ *is order-isomorphic to* $\mathbf{S}$*.*

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Let $\mathbf{S} = \langle S; \vee \rangle$ be a finite semilattice.

- For each covering pair $a \prec b$ in $\mathbf{S}$, define $f_{ba} \colon S \to S$ by

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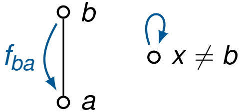

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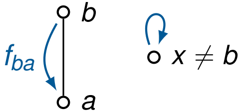

- Now define $F := \{ f_{ba} \mid a \prec b \text{ in } \mathbf{S} \}$ and $\mathbf{A} := \langle S; \vee, F \rangle$.

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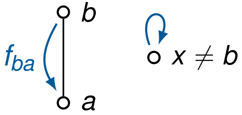
Let $\mathbf{S} = \langle S; \vee \rangle$ be a finite semilattice.

- For each covering pair $a \prec b$ in $\mathbf{S}$, define $f_{ba} \colon S \to S$ by

$$f_{ba}(x) := \begin{cases} a & \text{if } x = b, \\ x & \text{otherwise.} \end{cases}$$



- Now define $F := \{\, f_{ba} \mid a \prec b \text{ in } \mathbf{S} \,\}$ and $\mathbf{A} := \langle S; \vee, F \rangle$.
- Let $\downarrow\!\boldsymbol{x}$ denote the subalgebra of $\mathbf{A}$ with universe $\downarrow\! x$.

Interlude: a motivating example

Lemma (Birkhoff, Frink 1948)

For each finite join-semilattice $\mathbf{S}$*, there exists a finite algebra* $\mathbf{A}$ *such that* $\langle \mathrm{Sub}(\mathbf{A}); \leqslant \rangle$ *is order-isomorphic to* $\mathbf{S}$*.*

Proof.

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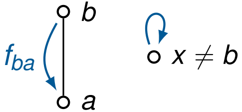
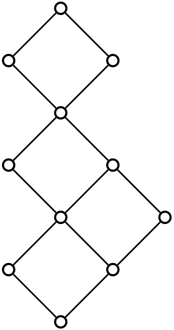


- Now define $F := \{ f_{ba} \mid a \prec b \text{ in } \mathbf{S} \}$ and $\mathbf{A} := \langle S; \vee, F \rangle$.
- Let $\downarrow\!\boldsymbol{x}$ denote the subalgebra of $\mathbf{A}$ with universe $\downarrow\!x$.
- Then $\alpha \colon \mathbf{S} \to \mathrm{Sub}(\mathbf{A})$, given by $\alpha(x) := \downarrow\!\boldsymbol{x}$, is an order-isomorphism. □

From **L** to the quasi-primal algebra **A**



L

1. Let **L** be a finite distributive lattice.

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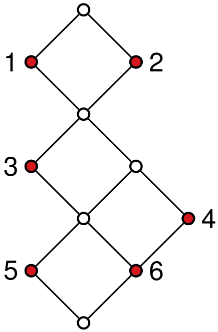

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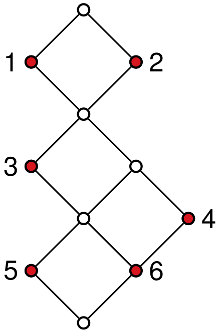

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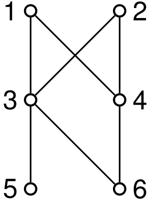

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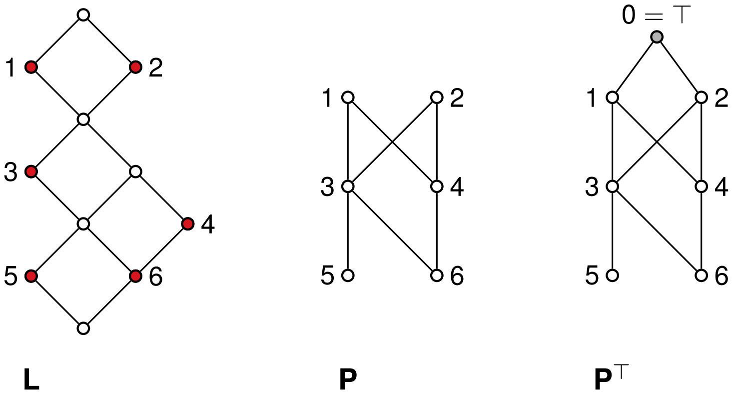
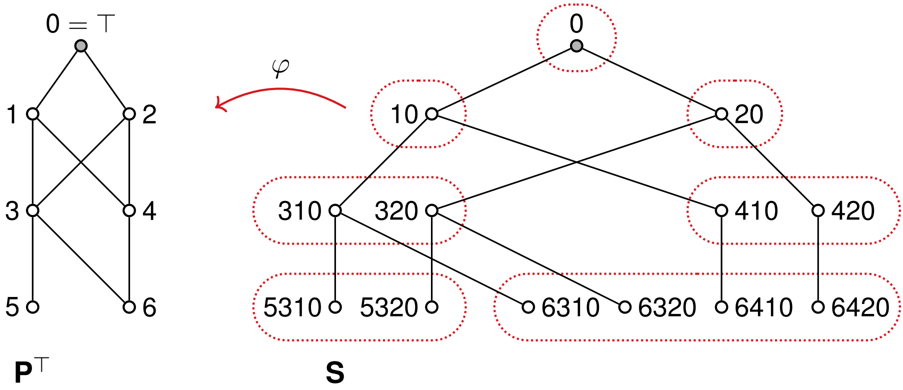


1. Let **L** be a finite distributive lattice.
2. Let $\mathbf{P} = \mathcal{J}(\mathbf{L})$.
3. Let $\mathbf{P}^\top$ denote **P** with a new top element $\top$ added.

From **L** to the quasi-primal algebra **A**

4. Let **S** be the universal covering tree of the ordered set $\mathbf{P}^\top$.



5. As a first approximation to **A**, let $\mathbf{A}_1 = \langle S; \vee, F_1, \tau \rangle$, where $F_1 := \{ f_{ba} \mid a \prec b \text{ in } \mathbf{S} \}$ and τ is the ternary discriminator.

From **L** to the quasi-primal algebra **A**

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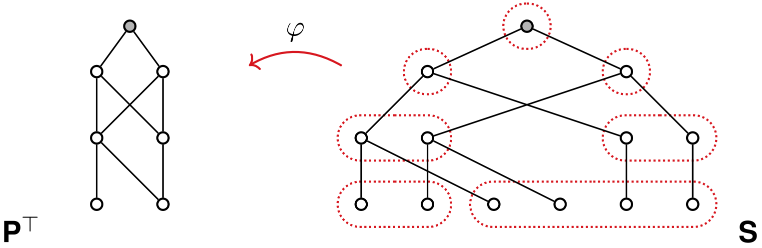

Bad: We have $\downarrow\boldsymbol{x} \ncong \downarrow\boldsymbol{y}$ for all $x \neq y$ in $S \setminus \mathrm{Min}(\mathbf{S})$,
but want $\downarrow\boldsymbol{x} \cong \downarrow\boldsymbol{y}$ whenever $\varphi(x) = \varphi(y)$.

From $\mathbf{L}$ to the quasi-primal algebra $\mathbf{A}$

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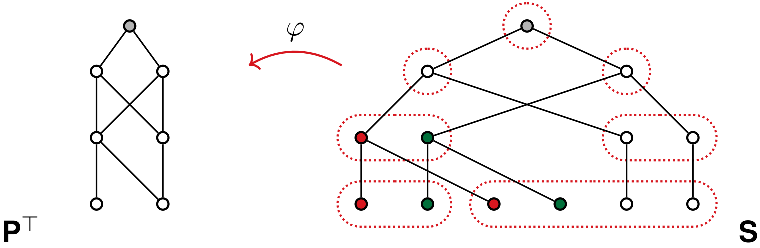

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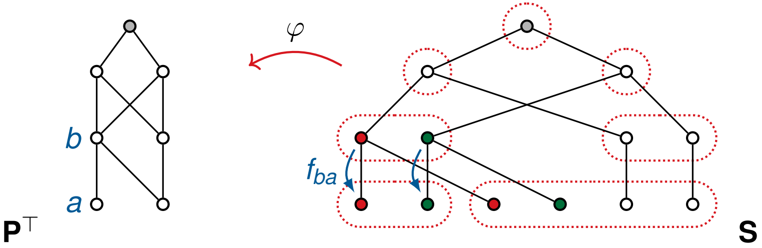

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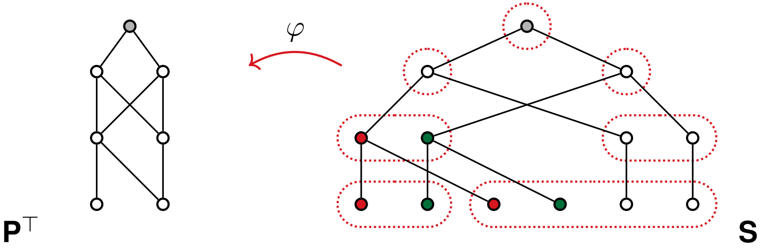
Fix: Redefine f_{ba} and index via covers $a \prec b$ in $\mathbf{P}^\top$.

From **L** to the quasi-primal algebra **A**

6. $\mathbf{A}_2 = \langle S; \vee, F_2, \tau \rangle$, where $F_2 := \{ f_{ba} \mid a \prec b \text{ in } \mathbf{P}^\top \}$ and τ is the ternary discriminator.



Good: We now have $\downarrow\boldsymbol{x} \cong \downarrow\boldsymbol{y}$ whenever $\varphi(x) = \varphi(y)$.

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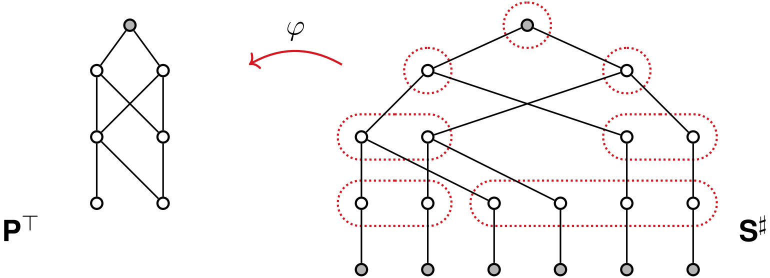

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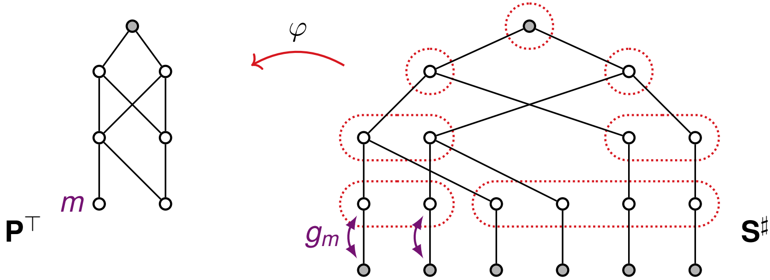

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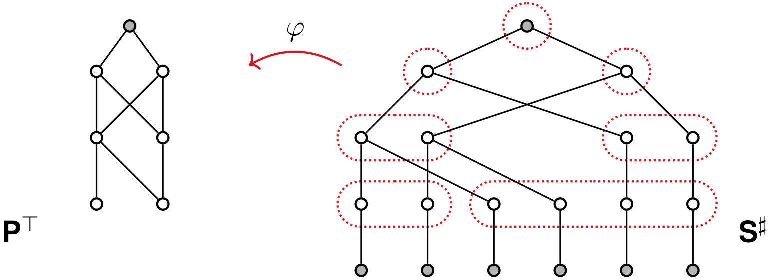
Fix: Let $\mathbf{S}^\sharp$ be $\mathbf{S}$ with every minimal element doubled. Then add operations g_m, indexed by $\mathrm{Min}(\mathbf{P}^\top)$, that flip the new and old minimal elements in an appropriate way.

From $\mathbf{L}$ to the quasi-primal algebra $\mathbf{A}$

7. $\mathbf{A}_3 = \langle S^\sharp; \vee, F_2, G, \tau \rangle$, where $F_2 := \{ f_{ba} \mid a \prec b \text{ in } \mathbf{P}^\top \}$, $G = \{ g_m \mid m \in \mathrm{Min}(\mathbf{P}^\top) \}$, τ is the ternary discriminator.



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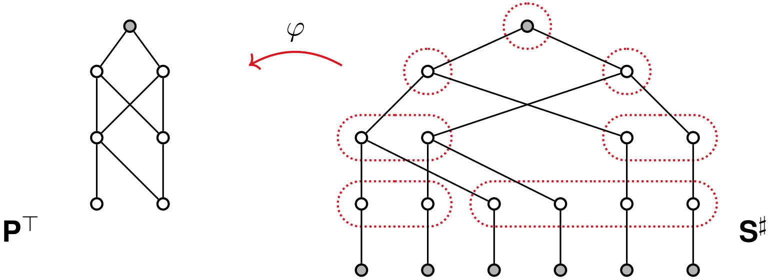

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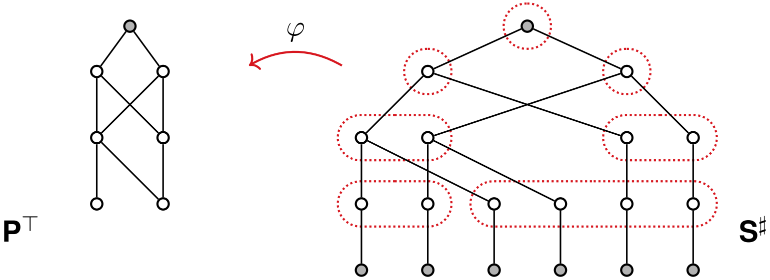

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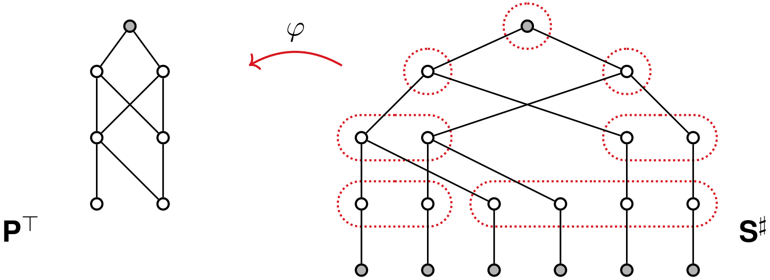

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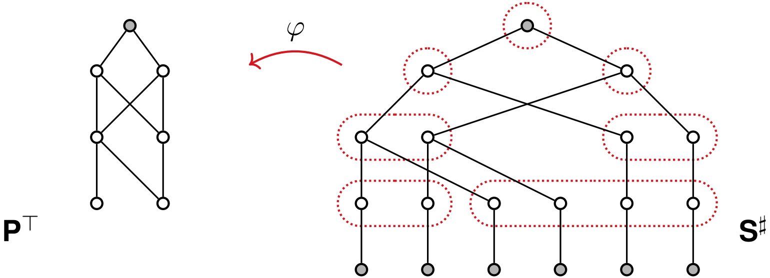
Fix: Remove all $f_{\top a}$ from F_2 and add an operation h so that

$(*)$ $\{\top\}$ is the only new subuniverse.

From $\mathbf{L}$ to the quasi-primal algebra $\mathbf{A}$

8. $\mathbf{A} = \langle S^\sharp; \vee, F_3, G, h, \tau \rangle$, where $F_3 := \{ f_{ba} \mid a \prec b \text{ in } \mathbf{P} \}$, $G = \{ g_m \mid m \in \mathrm{Min}(\mathbf{P}^\top) \}$, h has property $(*)$, and τ is the ternary discriminator.



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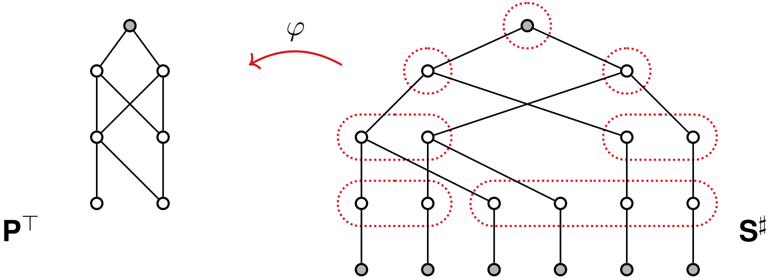

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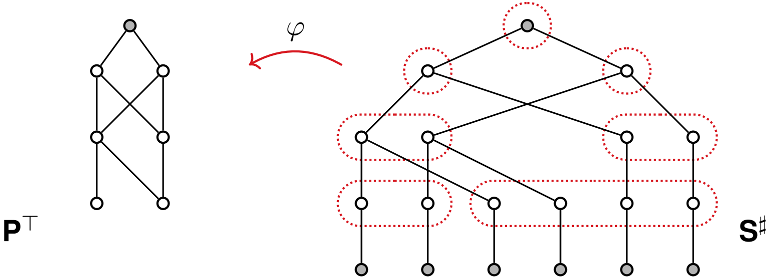

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Thus **A** is a quasi-primal algebra, with a trivial subalgebra, such that $\langle \mathrm{Sub}(\mathbf{A})/{\equiv}; \to\rangle \cong \mathbf{P}^\top$. Hence $\mathbf{L}_\mathbf{A} \cong \mathcal{O}(\overline{\mathbf{P}^\top}) = \mathcal{O}(\mathbf{P}) \cong \mathbf{L}$.

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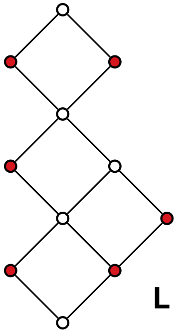

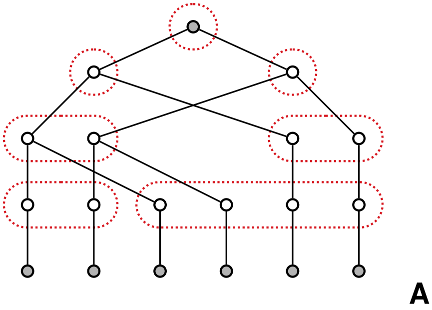

Theorem

For each finite distributive lattice **L**, *there exists a quasi-primal algebra* **A** *such that* $\mathbf{L_A}$ *is isomorphic to* **L**.

The class $\mathcal{L}_{\text{hom}}$

Notation

Let $\mathcal{L}_{\text{hom}}$ be the class of all lattices $\mathbf{L}$ such that $\mathbf{L} \cong \mathbf{L}_{\mathbf{A}}$, for some finite algebra $\mathbf{A}$.

Questions

- We wish to know which (finite) lattices belong to $\mathcal{L}_{\text{hom}}$.
- What general properties does the class $\mathcal{L}_{\text{hom}}$ have?

  For example:
  - We know $\mathcal{L}_{\text{hom}}$ is closed under finite products.
  - It is not clear whether $\mathcal{L}_{\text{hom}}$ is closed under forming homomorphic images or taking sublattices.

Congruence lattices

Finite Lattice Representation Problem

Does every finite lattice arises as the congruence lattice of a finite algebra?

- This is one of the most famous unsolved problems in universal algebra.
- It is therefore natural to ask whether the congruence lattice of each finite algebra belongs to $\mathcal{L}_{\mathrm{hom}}$.
- That is, given a finite algebra $\mathbf{A}$, does there exist a finite algebra $\mathbf{B}$ such that $\mathbf{L}_{\mathbf{B}} \cong \mathrm{Con}(\mathbf{A})$?
- We focus on the special case where $\mathbf{L}_{\mathbf{A}} \cong \mathrm{Con}(\mathbf{A})$.

Congruence lattices

- We can assume, without affecting the lattice Con($\mathbf{A}$), that every element of $\mathbf{A}$ is the value of a nullary term function.
- Then, for all $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$, the values of the nullary term functions of $\mathbf{B}$ form a subalgebra that we will denote by $\mathbf{B}_0$.

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Congruence Lattice Lemma

Let $\mathbf{A}$ be a finite algebra such that each element is the value of a nullary term function. Then the following are equivalent:

1. $\mathbf{L}_{\mathbf{A}} \cong \mathrm{Con}(\mathbf{A})$;
2. *for every finite algebra* $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$, *we have* $\mathbf{B} \to \mathbf{B}_0$;
3. a. $\mathbf{B} \to \mathbf{B}_0$, *for every finite subdirectly irreducible algebra* $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$, *and*
   b. $(\mathbf{A}/\theta_1) \times (\mathbf{A}/\theta_2) \to \mathbf{A}/(\theta_1 \cap \theta_2)$, *for all* $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$.

Subspace lattices and partition lattices

Corollary of (2) $\Rightarrow$ (1)

Let $\mathbf{A}$ *be a finite algebra.*

- *Assume that every finite algebra in* $\mathrm{Var}(\mathbf{A})$ *has a retraction onto each of its subalgebras.*
- *Let* $\mathbf{A}^+$ *be the algebra obtained from* $\mathbf{A}$ *by making each element of* $\mathbf{A}$ *the value of a nullary operation.*

Then $\mathbf{L}_{\mathbf{A}^+} \cong \mathrm{Con}(\mathbf{A})$, *whence* $\mathrm{Con}(\mathbf{A}) \in \mathcal{L}_{\mathrm{hom}}$.

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Applications of the corollary

- *Let* $\mathbf{V}$ *be a finite dimensional vector space over a finite field. The lattice of subspaces of* $\mathbf{V}$ *belongs to* $\mathcal{L}_{\mathrm{hom}}$.
- *Let* S *be a finite set. The lattice of equivalences on* S *belongs to* $\mathcal{L}_{\mathrm{hom}}$. *That is, the class* $\mathcal{L}_{\mathrm{hom}}$ *contains all finite partition lattices.*

Partition lattices

Theorem (Pudlák, Tůma 1980)
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- Since the variety of lattices is generated by its finite members, the only equations true in the class $\mathcal{L}_{\mathrm{hom}}$ are those true in all lattices.
- In fact, more is true. The only universal first-order sentences true in the class $\mathcal{L}_{\mathrm{hom}}$ are those true in all lattices.

Subgroup lattices

The following are equivalent [Pálfy, Pudlák 1980]:

- every finite lattice arises as the congruence lattice of a finite algebra;
- every finite lattice arises as an interval $[H, G]$ in the subgroup lattice of a finite group G.

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Application of the lemma (Keith Kearnes)

- *Let G be a group and let $\mathbf{A} = \langle A; \{\lambda_g \mid g \in G\}\rangle$ be a G-set, with at least one singleton orbit, regarded as a unary algebra. Then $\mathbf{L}_{\mathbf{A}^+} \cong \mathrm{Con}(\mathbf{A})$.*
- *If $A = G/H \cup \{\infty\}$ with the natural left action of G on cosets and with ∞ fixed, then $\mathbf{L}_{\mathbf{A}^+} \cong \mathrm{Con}(\mathbf{A}) \cong [H, G] \oplus \mathbf{1}$.*

The lattice $\mathbf{N}_5$

- With the exception of $\mathbf{N}_5$, we now know that all lattices of size up to 5 belong to $\mathcal{L}_{\mathrm{hom}}$.
- Our aim now is to give an example of a finite algebra $\mathbf{A}$ such that $\mathbf{L}_{\mathbf{A}} \cong \mathbf{N}_5$.

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- We shall apply (3) $\Rightarrow$ (1) of the Congruence Lattice Lemma.

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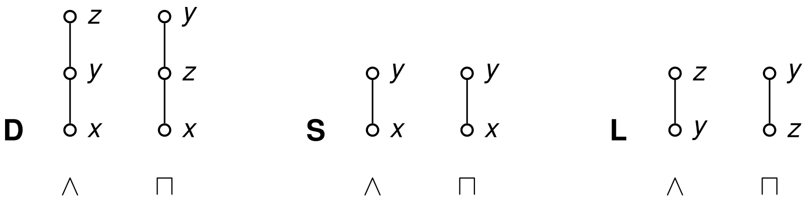
1. $\mathbf{L_A} \cong \mathrm{Con}(\mathbf{A})$;
3. a. $\mathbf{B} \to \mathbf{B}_0$, *for every finite subdirectly irreducible algebra* $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$, *and*
   b. $(\mathbf{A}/\theta_1) \times (\mathbf{A}/\theta_2) \to \mathbf{A}/(\theta_1 \cap \theta_2)$, *for all* $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$.

Distributive bisemilattices

- An algebra $\mathbf{A} = \langle A; \wedge, \sqcap \rangle$ is a distributive bisemilattice if both $\wedge$ and $\sqcap$ are semilattice operations on A and each distributes over the other.



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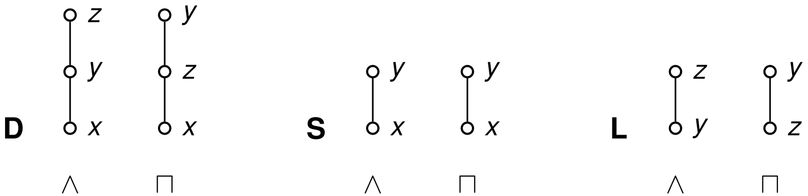

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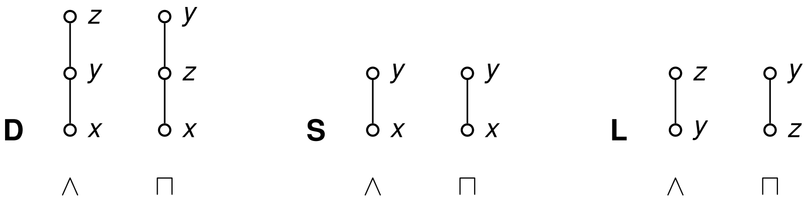

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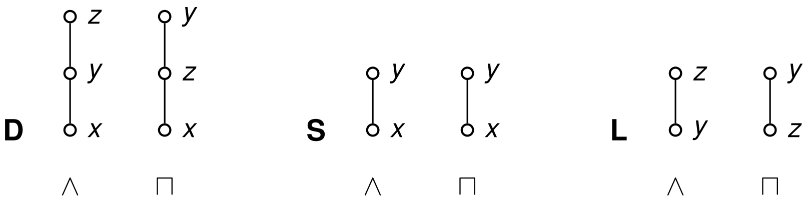
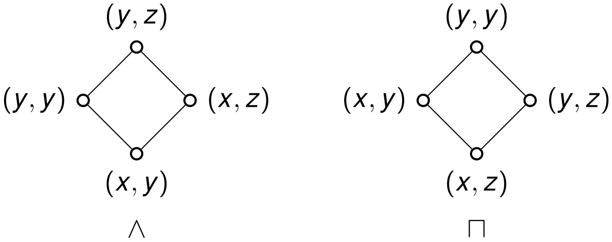
  - **L** is the two-element lattice.



The lattice $\mathbf{N}_5$

Example

Let $\mathbf{A} = \langle\{0, a, b, 1\}; \wedge, \sqcap, 0, a, b, 1\rangle$ *be the distributive bisemilattice* $\mathbf{S} \times \mathbf{L}$ *with all four elements added as nullaries. Then* $\mathbf{L}_{\mathbf{A}} \cong \mathbf{N}_5$*.*



The distributive bisemilattice $\mathbf{S} \times \mathbf{L}$

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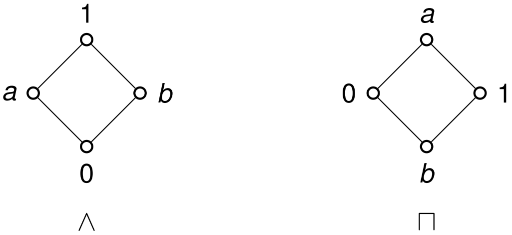

The algebra **A**

The lattice $\mathbf{N}_5$

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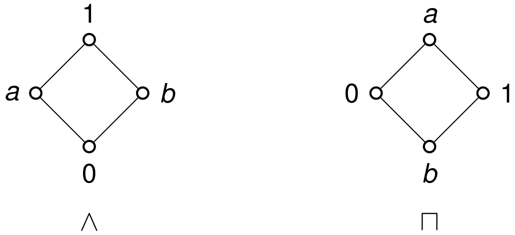
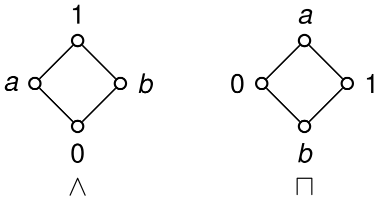


The algebra **A**

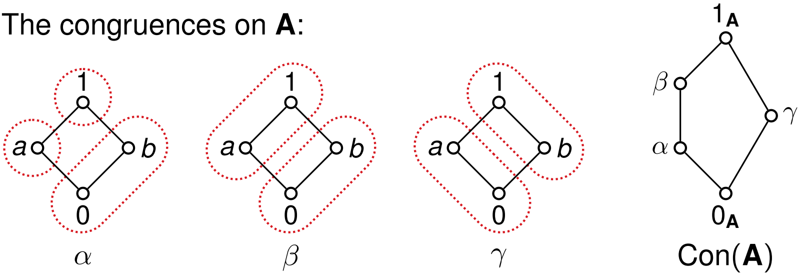
- The bisemilattice $\mathbf{S} \times \mathbf{L}$ is a reduct of the 4-element distributive bilattice made famous by Belnap (1977).

Proof of the $\mathbf{N}_5$ example

- The algebra $\mathbf{A}$:



- The congruences on $\mathbf{A}$:



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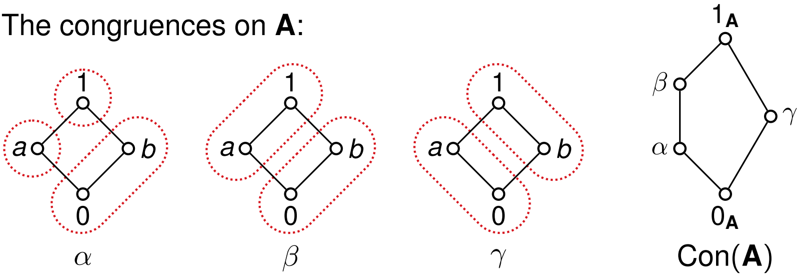
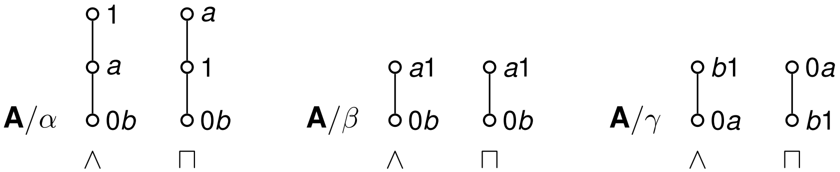
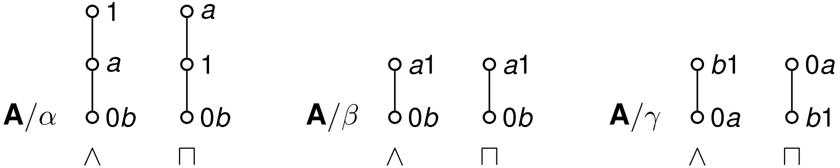


- The corresponding quotients:



▸ Skip proof

Proof of the $\mathbf{N}_5$ example



- Since α, β and γ are meet-irreducible in Con(**A**), it follows that **A**$/\alpha$, **A**$/\beta$ and **A**$/\gamma$ are subdirectly irreducible algebras in Var(**A**).

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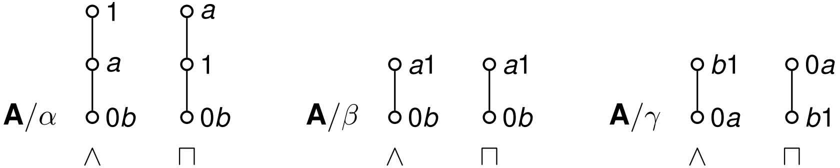

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- We claim that, up to isomorphism, these are the only subdirectly irreducible algebras in Var(**A**).

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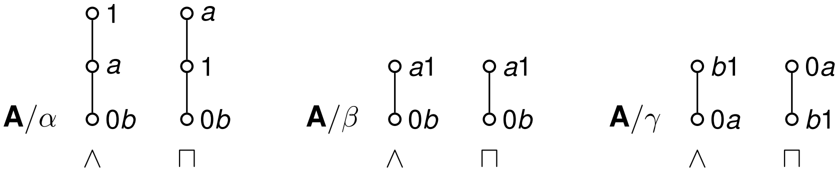

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- Note that each of these algebras satisfies $\mathbf{B} = \mathbf{B}_0$.

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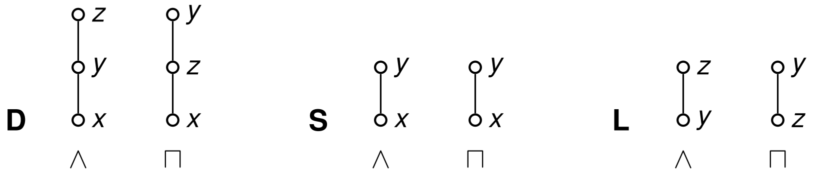

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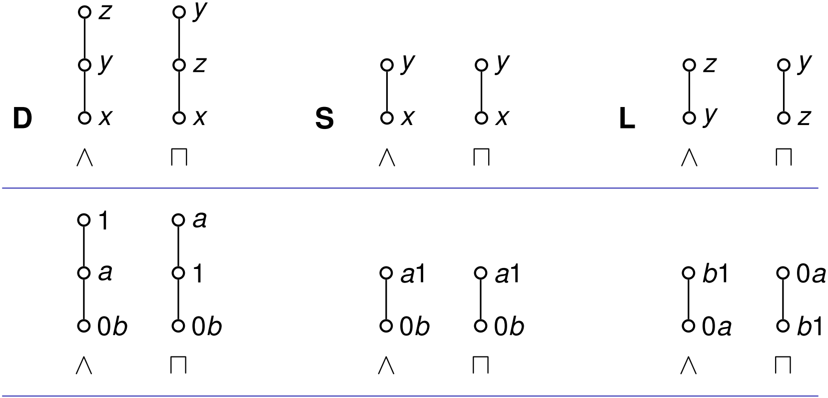

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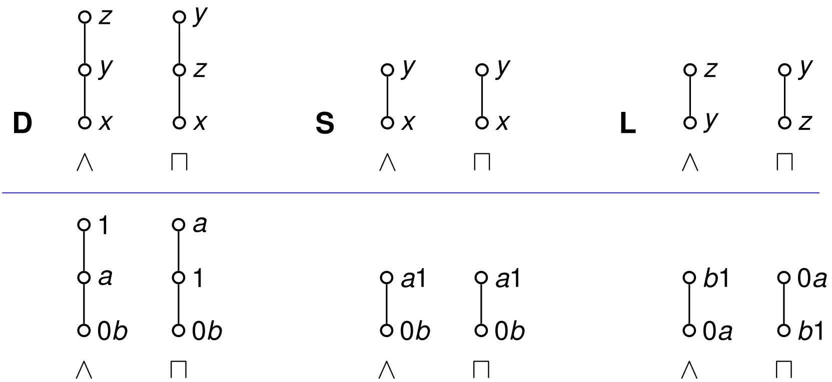
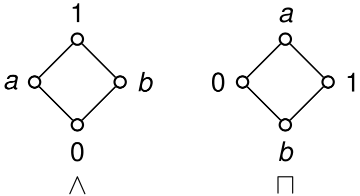


- 0 is the bottom for $\wedge$
- 1 is the top for $\wedge$
- a is the top for $\sqcap$
- b is the bottom for $\sqcap$



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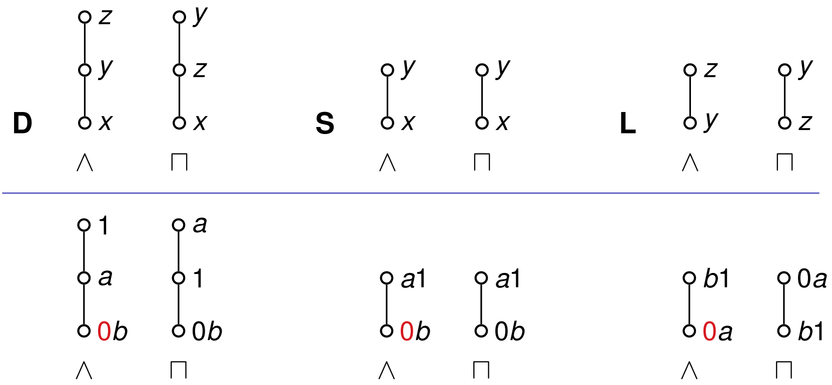

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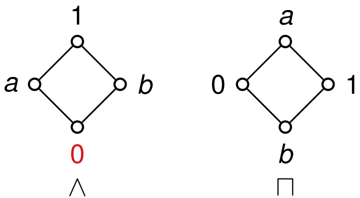

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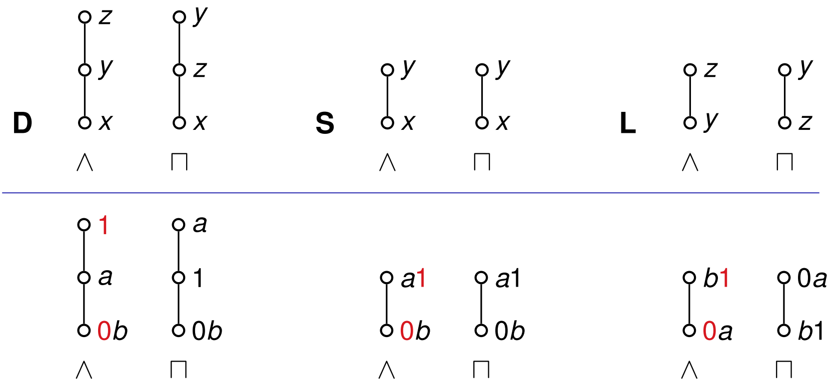

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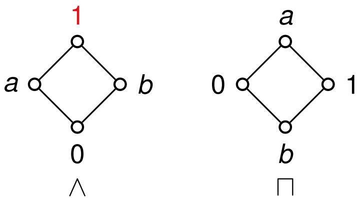

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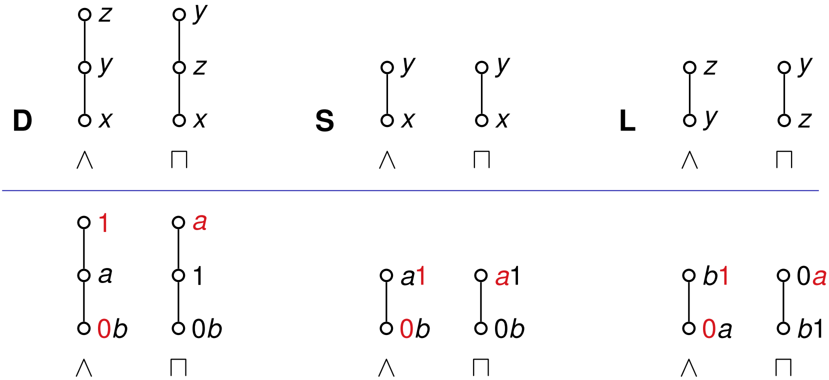

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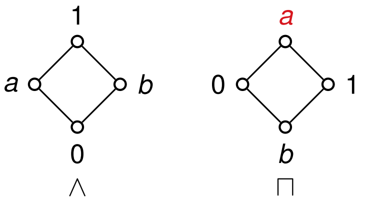

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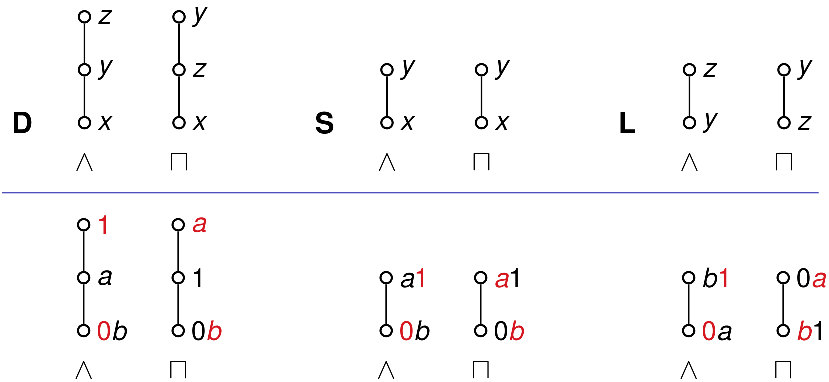

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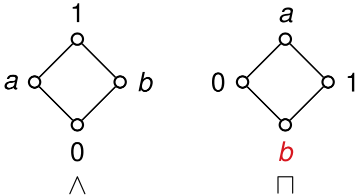

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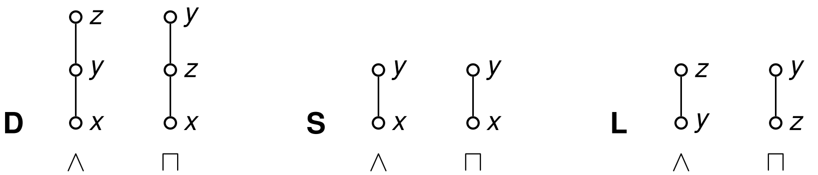

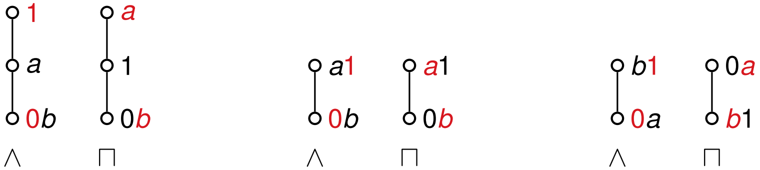

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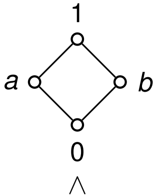

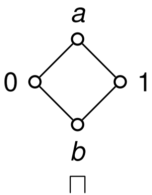

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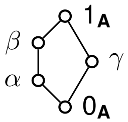

Con($\mathbf{A}$)

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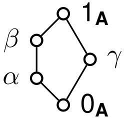

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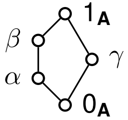

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- Hence $\mathbf{L}_{\mathbf{A}} \cong \mathrm{Con}(\mathbf{A}) \cong \mathbf{N}_5$, by the Congruence Lattice Lemma. $\square$

Problems

Representation

- Does every countable bounded lattice belong to $\mathcal{L}_{\mathrm{hom}}$?
- In particular, does $\mathcal{L}_{\mathrm{hom}}$ contain:
  - every finite lattice?
  - the congruence lattice of every finite algebra?
  - every interval in the subgroup lattice of a finite group?
- Is every finite homomorphism lattice $\mathbf{L_A}$ representable as the congruence lattice of a finite algebra?

Finiteness

- For which finite algebras $\mathbf{A}$ is the lattice $\mathbf{L_A}$ finite? Is this decidable?

Classes of algebras

- Restrict to natural classes of algebras. For example: unary algebras, semigroups with constants.