A Dichotomy for First-order Reducts of Unary Structures

Manuel Bodirsky

Joint work with Antoine Mottet.
Institut für Algebra, TU Dresden

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Happy Birthday!

Working assumption

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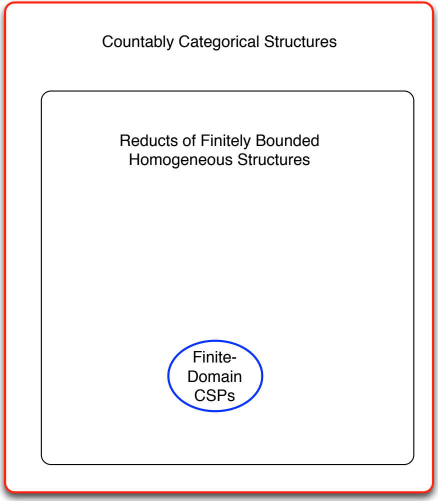
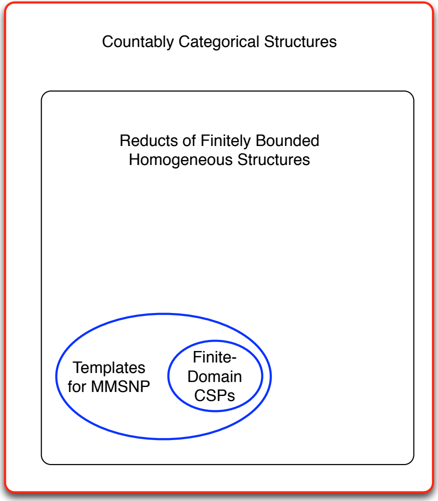
Can we go home?

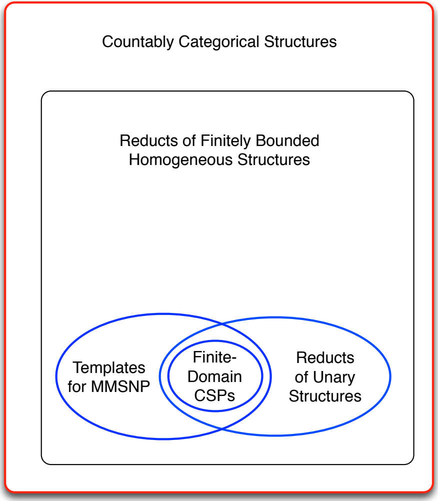
The current landscape



Feder and Vardi



What next?


Countably Categorical Structures
Reducts of Finitely Bounded Homogeneous Structures
Templates for MMSNP
Finite-Domain CSPs
Reducts of Unary Structures

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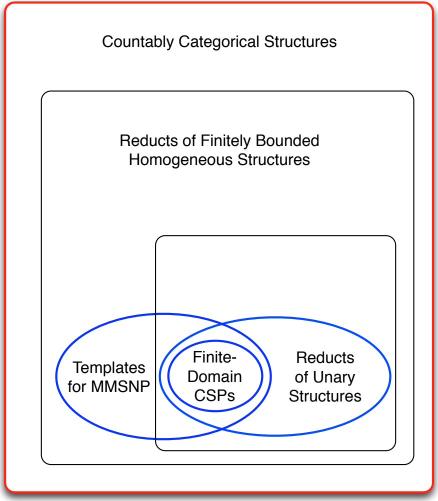

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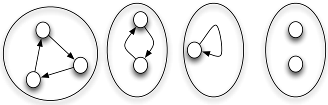

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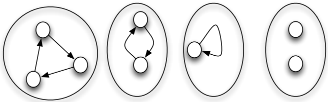

Fact: If Γ is a finitely bounded structure, then CSP(Γ) is in NP.

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For many classes of structures Γ there a polynomial-time reduction in the other direction, from CSP(T_Γ) to CSP(Γ)!

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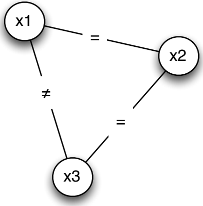

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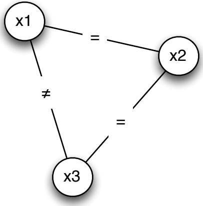

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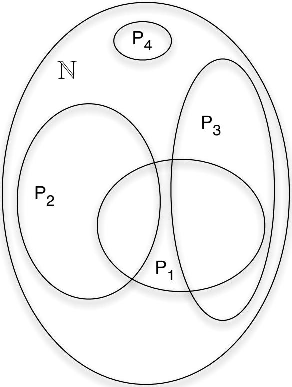

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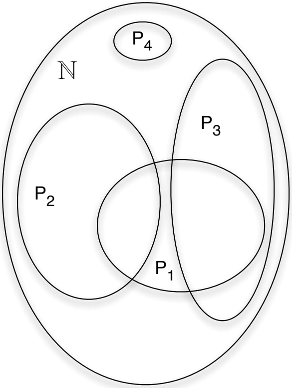

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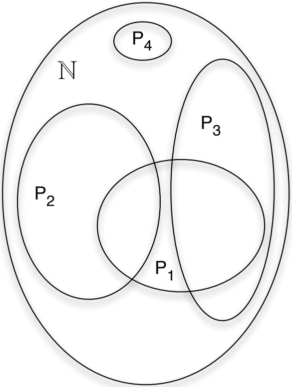

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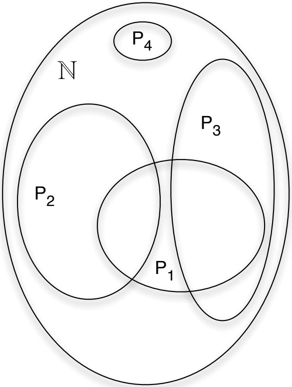

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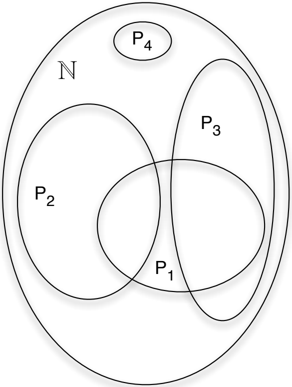
- 1-1 correspondence between (quantifier-free) m-types of $\mathfrak{U}$ and orbits of k-tuples wrt. $\mathrm{Aut}(\mathfrak{U})$.
- First-order reducts of unary structures are ω-categorical: they are up to isomorphism given by their first-order theory (equivalently: $\mathrm{Aut}(\Gamma)$ is oligomorphic).

First-order Reducts of Unary Structures

Let $\mathfrak{U} = (\mathbb{N}; P_1, \dots, P_n)$ be a structure
where $P_1, \dots, P_n$ are unary.
Let $\Gamma = (\mathbb{N}; R_1, \dots, R_m)$ be a structure
where $R_1, \dots, R_m$ are first-order definable in $\mathfrak{U}$.
Γ called first-order reduct of a unary structure.



Facts.

- $\mathfrak{U}$ is finitely bounded.
- 1-1 correspondence between (quantifier-free) m-types of $\mathfrak{U}$ and orbits of k-tuples wrt. $\mathrm{Aut}(\mathfrak{U})$.
- First-order reducts of unary structures are ω-categorical: they are up to isomorphism given by their first-order theory (equivalently: $\mathrm{Aut}(\Gamma)$ is oligomorphic).
- Every finite structure is homomorphically equivalent to a first-order reduct of a unary structure.

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- If Γ is ω-categorical, the complexity of CSP(Γ) is captured by Pol(Γ) as a topological clone (MB+Pinsker'15).

Reducts of Unary Structures: Preparations

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Every ω-categorical structure Γ is homomorphically equivalent to an ω-categorical structure Δ such that $\mathrm{End}(\Delta) = \overline{\mathrm{Aut}(\Delta)}$.

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Lemma. The model-complete core of a first-order reduct of a unary structure is again a first-order reduct of a unary structure.

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$$e_1 f(y, x, \ldots, x, x) = e_2 f(x, y, \ldots, x) = \cdots = e_n f(x, \ldots, x, y)$$

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Definition. A function $f\colon B^n \to B$ is called k-canonical (wrt. Γ) iff for all $t_1, \dots, t_n \in B^k$ the orbit of $f(t_1, \dots, t_n)$ wrt. $\mathrm{Aut}(\Gamma)$ only depends on the orbits of $t_1, \dots, t_n$ wrt. $\mathrm{Aut}(\Gamma)$.

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Lemma (MB,Pinsker'11).

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contains a function g that is canonical wrt Γ.

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4. Can be shown that then Γ also contains canonical cyclic polymorphism.

Pol(Γ)

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Reference. A Dichotomy for First-Order Reducts of Unary Structures, MB and Antoine Mottet, 2017.
https://arxiv.org/pdf/1601.04520.pdf
A subset of the results was announced at LICS'16.