Critical points for congruence lattices

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June 15, 2017

Congruences

Congruence on algebra $\mathbf{A}$ is an equivalence relation θ on the set A, preserved by all basic operations of $\mathbf{A}$, i.e.

$$(a_1, b_1) \in \theta,\ (a_2, b_2) \in \theta, \ldots, (a_n, b_n) \in \theta$$

implies

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \in \theta))$$

(for f n-ary).
Every algebra with more than 1 element has at least two congruences.

Importance of congruences

Congruences enable *quotients*:
On the set of θ-classes we define the operations be means of representatives:

$$f(a_1/\theta, \ldots, a_n/\theta) = f(a_1, \ldots, a_n)/\theta.$$

This gives rise to a new algebra of the same type as $\mathbf{A}$, which is a simplified image of the algebra A.

For instance, $\mathbb{Z}/(\operatorname{mod} n) = \mathbb{Z}_n$.

Congruence lattices

Congruences on an algebra $\mathbf{A}$ can be ordered by the “refinement" relation (= set inclusion):

$$\varphi \leq \theta \text{ ak } (x\varphi y \text{ implies } x\theta y).$$

We obtain an algebraic lattice Con$\mathbf{A}$.
Conversely,

Theorem

(G. Grätzer, E. T. Schmidt) *A lattice is isomorphic to the congruence lattice of some algebra if and only if it is algebraic.*

Algebraic lattices

An element a of a lattice L is called *compact* if for every $K \subseteq L$

$$a \leq \sup K \quad \text{implies} \quad a \leq \sup K_0$$

for some finite subset $K_0 \subseteq K$.
The set of all compact elements is a $\vee$-subsemilattice of L, denoted by L_c. The lattice L is compact if it is complete and

$$x = \sup\{a \in L_c \mid a \leq x\}$$

for every $x \in L$.
Fact: Every algebraic lattice L is isomorphic to $\mathrm{Id}(L_c)$, the ideal lattice of L_c.

General problem

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Very few relevant classes with a satisfactory answer

Theorem

L is isomorphic to $\mathrm{Con}\, B$ for some Boolean algebra B iff L_c is a Boolean lattice.

Theorem

L is isomorphic to $\mathrm{Con}\, D$ for some distributive lattice D iff L_c is a generalized Boolean lattice.

Theorem

L is isomorphic to $\mathrm{Con}\, A$ for some Stone algebra A iff L_c is a dual Stone lattice.

Congruence lattices

One solved problem: Is every *distributive* algebraic lattice (isomorphic to) the congruence lattice of some lattice?

Partial positive results (R. P. Dilworth, E. T. Schmidt, A. Huhn...), but
Final answer: no (F. Wehrung 2005)

For most common classes of algebras, a full description of $\operatorname{Con} \mathcal{K}$ seems hopeless.

A more tractable problem:

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
(Con $\mathcal{K} \subseteq$ Con $\mathcal{L}$)

and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

Some critical points

We are especially interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are congruence-distributive varieties (in most results also finitely generated). For instance,
$\mathrm{Crit}(\mathbf{N}_5, \mathbf{M}_3) = 5$,
$\mathrm{Crit}(\mathbf{M}_3, \mathbf{N}_5) = \mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) = \aleph_0$,
$\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$,
$\mathrm{Crit}(\mathbf{Maj}, \mathbf{Lat}) = \aleph_2$.
($\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{M}_4$, $\mathbf{D}$ are well-known lattice varieties, $\mathbf{Lat}$ = all lattices, $\mathbf{Maj}$ = all majority algebras.)
P. Gillibert: under some reasonable finiteness conditions, the critical point between two varieties cannot be larger than $\aleph_2$.

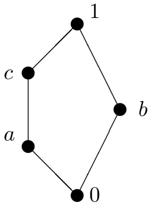
Why L_c?

$\mathrm{Con}_c A$ reflects the size of A better.

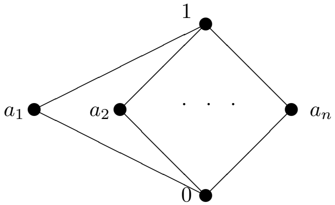
Theorem

If an infinite algebra A is a subdirect product of finite algebras of bounded size, then $|\mathrm{Con}_c A| = |A|$.

N_5 and M_n



N_5



M_n

No $\aleph_3$?

Theorem

(Gillibert)
Let $\mathcal{V}$ and $\mathcal{W}$ be locally finite varieties. Assume that for any finite $L \in \operatorname{Con}\mathcal{V}$ there are, up to isomorphism, finitely many $B \in \mathcal{W}$ with $L \cong \operatorname{Con} B$, and each such B is finite. Then $\operatorname{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_2$ or $\operatorname{Con}\mathcal{V} \subseteq \operatorname{Con}\mathcal{W}$.

Any finitely generated congruence-distributive varieties satisfy the assumptions.

Possible cases

For finitely generated congruence-distributive varieties there are following possible cases:

- $\mathrm{Crit}(\mathcal{V}, \mathcal{W})$ is finite;
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) = \aleph_0$;
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) = \aleph_1$;
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) = \aleph_2$;
- $\mathrm{Con}\,\mathcal{V} \subseteq \mathrm{Con}\,\mathcal{W}$.

How to distinguish?

Topological approach

$\mathrm{M}(L)$....completely meet-irreducible elements of a lattice L
($a = \inf X$ implies $a \in X$)
Fact: if L is algebraic, then every element is a meet of completely meet-irreducible elements.
Topology on $\mathrm{M}(L)$: all sets of the form

$$\mathrm{M}(L) \cap \uparrow x = \{a \in \mathrm{M}(L) \mid a \geq x\}$$

are closed.

Theorem

If L is distributive algebraic, then $L \cong \mathcal{O}(\mathrm{M}(L))$. (The lattice of all open subsets of $\mathrm{M}(L)$.

Topological approach

Sometimes the properties of $\mathrm{Con}\, A$ are more effectively expressed as topological properties of $\mathrm{M}(\mathrm{Con}\, A)$. A sample:

- If $A \in \mathbf{D}$ then $\mathrm{M}(\mathrm{Con}\, A)$ is Hausdorff.
- There exists a countable $B \in \mathbf{M}_3$ such that $\mathrm{M}(\mathrm{Con}\, B)$ is not Hausdorff.
- Therefore, $\mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) \leq \aleph_0$.

The topological approach was used to establish e.g. $\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$. (But the argument is much more complicated.)

How much more complicated?

A topological space T is called n-uniformly separable ($n \geq 3$) if for every discrete subset Q there is a family $\{B_{pq} \mid p, q \in Q,\ p \neq q\}$ of open sets such that $p \in B_{pq}$ and $\bigcap\{B_{pq} \mid p, q \in Q_0\} = \emptyset$ whenever $Q_0 \subseteq Q$, $|Q_0| \geq n$.

- $\mathrm{M}(\mathrm{Con}\, A)$ is $(n+1)$-uniformly separable for every $A \in \mathbf{M}_n$.
- $\mathrm{M}(\mathrm{Con}\, F_n(X))$ is not n-uniformly separable.
- Therefore, $\mathrm{Crit}(\mathbf{M}_{n+1}, \mathbf{M}_n) \leq \aleph_2$.

The proof of $\mathrm{Crit}(\mathbf{M}_{n+1}, \mathbf{M}_n) \geq \aleph_2$: a tedious transfinite induction

Con functor

The Con functor:

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}\, f: \mathrm{Con}\, A \to \mathrm{Con}\, B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}\, f$ preserves $\vee$ and 0, not necessarily $\wedge$.

Lifting of semilattice morphisms

Let

- $\varphi: S \to T$ be a $(\vee, 0)$-homomorphisms of lattices;
- $f: A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1: S \to \operatorname{Con} A$, $\psi_2: T \to \operatorname{Con} B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ {\scriptstyle \psi_1}\downarrow & & {\scriptstyle \psi_2}\downarrow \\ \operatorname{Con} A & \xrightarrow{\operatorname{Con} f} & \operatorname{Con} B \end{array}$$

commutes.

A generalization: lifting of semilattice diagrams

Results of P. Gillibert

Theorem

Let $\mathcal{V}$, $\mathcal{W}$ be finitely generated congruence distributive varieties. TFAE

- $\operatorname{Con}\mathcal{V} \nsubseteq \operatorname{Con}\mathcal{W}$;
- *there exists a diagram of finite $(\vee, 0)$-semilattices indexed by a n-dimensional cube (for some n) liftable in $\mathcal{V}$ but not in $\mathcal{W}$*

Results of P. Gillibert

Theorem

Let $\mathcal{V}$, $\mathcal{W}$ be finitely generated congruence distributive varieties. Then (2) implies (1), where

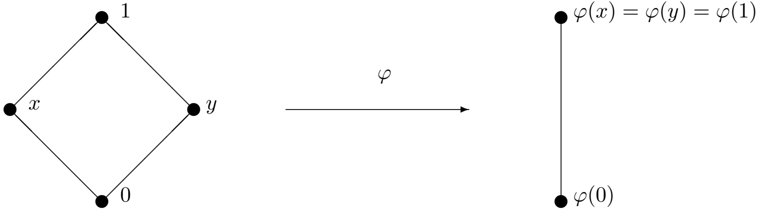
- $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_n$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a product of* $n+1$ *finite chains liftable in* $\mathcal{V}$ *but not in* $\mathcal{W}$

If $n = 0$ *then also (1)*$\Longrightarrow$ *(2).*

Question. What about (1)$\Longrightarrow$(2) for $n > 0$? (Conjecture: not true.)

Example

The semilattice homomorphism



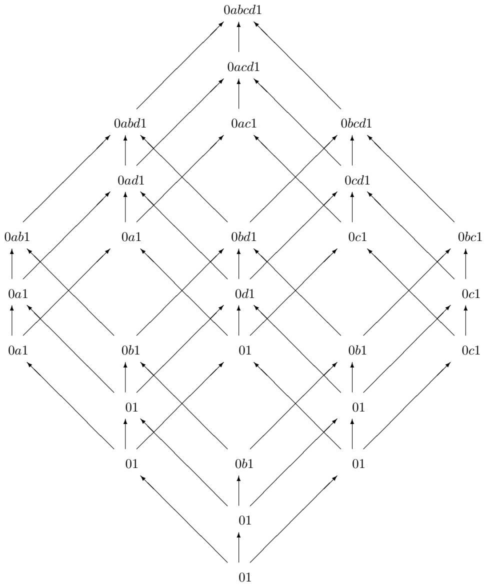
has a lifting in $\mathbf{M}_3$ (the embedding of a 3-element chain into M_3 lifts it), but not in $\mathbf{D}$. Therefore, $\mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) \leq \aleph_0$.

Diagram lifting and aleph2

We know that $\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$. Is there a diagram indexed by a product of 3 finite chains liftable in $\mathbf{M}_4$ but not in $\mathbf{M}_3$?

Yes, it is on the next slide. It is an alternative proof of the inequality $\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$.

M3 versus M4


0abcd1
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0d1
0c1
0a1
0b1
01
0b1
0c1
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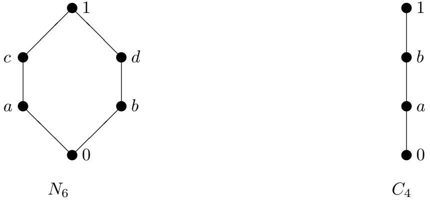
01

Critical point $\aleph_1$

Let $\mathbf{C}_4^*$ and $\mathbf{N}_6^*$ be the varieties generated by the bounded lattices C_4 and N_6 with an additional unary operation:
on C_4 ... $f(0) = 0$, $f(a) = b$, $f(b) = a$, $f(1) = 1$;
on N_6 ... $180°$ rotation ($f(a) = d$...) .



Critical points $\aleph_1$

Theorem

(1) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{N}_5) = \aleph_1$;

(2) $\mathrm{Crit}(\mathbf{N}_5, \mathbf{N}_6^*) = \aleph_0$.

(3) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{C}_4^*) = \aleph_1$;

(4) $\mathrm{Crit}(\mathbf{C}_4^*, \mathbf{N}_6^*) = \infty$.

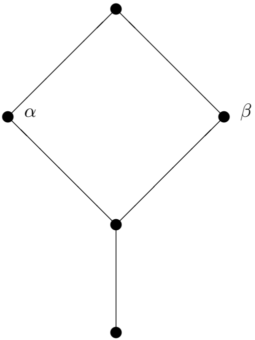
(N_5 considered as bouned lattice.)

Question

What is the mechanism behind these examples?

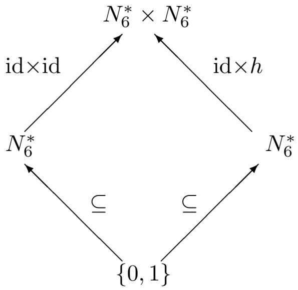
N6 versus N5

Both N_5 and N_6^* have the same congruence lattice, but N_6^* has an automorphism h (the vertical symmetry), such that $\operatorname{Con}_c h$ interchanges α and β:



N6 versus N5

Below: $\mathcal{D}$ is the diagram in $\mathbf{N}_6^*$, so that $\mathrm{Con}\,\mathcal{D}$ has a lifting in $\mathbf{N}_6^*$ but - no lifting in $\mathbf{N}_5$.



Possible variations

We can construct more examples by considering semilattice morphisms with several liftings in one of the classes and combining them in square diagrams. But we have to be careful to ensure critical point not $\leq \aleph_0$

Looking for a link

What is the connection between liftability of diagrams and topological properties of dual spaces? We have an answer in the following special case.
We say that a variety $\mathcal{K}$ has *Compact Congruence Intersection Property (CCIP)* if $\mathrm{Con}_c\, A$ is a lattice for every $A \in \mathcal{K}$.

Examples: distributive lattices, Stone algebras, vector spaces ...

Theorem

(Baker, Blok, Pigozzi) *A finitely generated congruence-distributive variety $\mathcal{K}$ has CCIP iff every subalgebra of a subdirectly irreducible algebra in $\mathcal{K}$ is subdirectly irreducible (or one-element).*

CCIP

CCIP greatly simplifies the investigation of congruence lattices. (But it is still difficult.)

- Every $\mathcal{V}$ with $\operatorname{Con}\mathcal{V}$ satisfactorily described has CCIP.
- Every distributive algebraic lattice whose compact elements form a lattice is the congruence lattice of
  (a) a lattice (E. T. Schmidt);
  (b) a locally matricial algebra (P. Ružička).

Priestley duality

Let A belong to a finitely generated congruence-distributive variety with CCIP. Then $\mathrm{Con}_c\, A$ is a distributive lattice and we can consider its dual Priestley space $\mathbf{X}$. (We use the version for lattices with 0 but not necessarily with 1.)

- Points of the dual space correspond to (completely) meet-irreducible elements of $\mathrm{Con}\, A$, so $\mathbf{X} = \mathrm{M}(\mathrm{Con}\, A)$. (Here we declare $1 \in \mathrm{M}(\mathrm{Con}\, A)$.)
- The order is inherited from $\mathrm{Con}\, A$.
- The topology is generated by the sets

$$M_{x,y} = \{\alpha \mid (x, y) \in \alpha\}$$

  and their complements.

Diagrams versus topology

Theorem

Let $\mathcal{V}$ be a finitely generated congruence-distributive variety with CCIP. Let F be the free algebra in $\mathcal{V}$ with $\aleph_1$ generators. Let $\vec{S}$ be a diagram of finite distributive $(0, \vee)$-semilattices indexed by a finite ordered set P having a smallest element $0 \in P$. The following conditions are equivalent.

(i) *There exists $A \in \mathcal{V}$ such that $\mathbf{X} = \mathrm{M}(\mathrm{Con}\, A)$ is $\vec{S}$-nonseparable;*

(ii) $\mathrm{M}(\mathrm{Con}\, F)$ *is $\vec{S}$-nonseparable;*

(iii) $\vec{S}$ *has a lifting in $\mathcal{V}$.*

What is $\vec{S}$-separability?

First,a simple example.
Let $\vec{S}$ be a diagram of $(\vee, 0)$-semilattices consisting of a single morphism $\varphi : \{0\} \to \{0, 1\}$.

- $\vec{S}$ in not liftable in the variety of Boolean algebras. Consequence: the largest element in the dual space (corresponding to 1-element congruence lattice) does not belong to the topological closure of the rest (points corresponding to 2-element lattices).
- $\vec{S}$ is liftable in the variety of distributive lattices. Consequence: the largest element in the dual space (corresponding to 1-element congruence lattice) *can* belong to the topological closure of the rest (points corresponding to 2-element lattices).

Definition in general

Let $\vec{S} = (S_p, s_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of finite distributive lattices and 0-preserving lattice homomorphisms, indexed by a finite ordered set P. Let $s^{\leftarrow}_{p,q}$ be the residuated maps, $s^{\leftarrow}_{p,q}(\alpha) = \sup\{\beta \mid s^{\leftarrow}_{p,q}(\beta) \leq \alpha\}$.
Let X be a Priestley space. For every $p \in P$ let K_p be the set of all order embeddings $\mathrm{M}(S_p) \to X$ whose range is an upper subset of X. An open neighbourhood of $k \in K_p$ is a family of open sets $U = (U(\alpha) \mid \alpha \in \mathrm{M}(S_p))$ with $k(\alpha) \in U(\alpha)$.
Definition. We say that X is $\vec{S}$-nonseparable, if for any system of open neighbourhoods U_k $(k \in \cup K_p)$ there exists a family $(k_p \in K_p \mid p \in P)$ such that $k_q(\alpha) \in U_{k_p}(s^{\leftarrow}_{p,q}(\alpha))$ whenever $p < q$, $\alpha \in \mathrm{M}(S_q)$. Otherwise we say that X is $\vec{S}$-separable.

Critical point for CCIP

Theorem

Let $\mathcal{V}$ be a finitely generated congruence-distributive variety with CCIP. Then $\mathrm{Con}\,\mathcal{V} \subseteq \mathrm{Con}\,\mathcal{W}$ *or* $\mathrm{Crit}(\mathcal{V}, \mathcal{W}) \leq \aleph_1$.

Proof: Let $\mathrm{Con}\,\mathcal{V} \nsubseteq \mathrm{Con}\,\mathcal{W}$. By Gillibert, there exist a diagram $\vec{S}$ indexed by some n-dimensional cube liftable in $\mathcal{V}$ but not in $\mathcal{W}$. Let F be the free algebra in $\mathcal{V}$ with $\aleph_1$ generators. By our theorem, $\mathrm{M}(\mathrm{Con}\,F))$ is $\vec{S}$-nonseparable, while $\mathrm{M}(\mathrm{Con}\,A)$ is $\vec{S}$-separable for every $A \in \mathcal{W}$. Consequently, $\mathrm{Con}\,F \notin \mathrm{Con}\,\mathcal{W}$ and the cardinality of F is $\aleph_1$.

Upper bound can occur: $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{C}_4^*) = \aleph_1$;

How $\aleph_1$ got there

Well known free set theorem:

Theorem

(Hajnal) *If* $|X| \geq \aleph_1$, *then for every function* $\Phi : X \to [X]^{<\omega}$ *there is a set* $Y \subseteq X$ *such that* $|Y| = |X|$ *and* $x \notin \Phi(y)$ *whenever* $x, y \in Y$, $x \neq y$.

Free set theorem for diagrams

Let $(A_p \mid p \in P)$ be a family of nonempty finite sets, indexed by a finite poset P with a smallest element, such that $A_p \subseteq A_q$ whenever $p \leq q$. For a set X let $H(X, A_p)$ denote the set of all *surjective* mappings $X \to A_p$

Theorem

If $|X| \geq \aleph_1$, then for every function

$$\mathrm{supp} : \bigcup_{p \in P} H(X, A_p) \to [X]^{<\omega}$$

there are $h_p \in H(X, A_p)$ such that

$$h_q \restriction \mathrm{supp}\, h_p = h_p \restriction \mathrm{supp}\, h_p$$

for every $p < q$.

(A diagram version of the free set theorem.)

Open problems

1. See Chapter 9 in G. Grätzer, F. Wehrung (eds.), *Lattice Theory: Special Topics ans Applications*, Birkhäuser 2014. Especially,
   - Is every distributive algebraic lattice isomorphic to the congruence lattice of a majority algebra?
2. Investigate classes Cw $\mathcal{K}$ (weak congruence lattices).

Thanks

Thank you for attention.

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