# Cube Terms and the Subpower Membership Problem

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  - lattices; more generally, algebras with NU terms [Baker–Pixley, 1975].

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- **Example 1.** $\mathcal{V}$ is a variety of groups with additional operations iff $\mathcal{V}$ satisfies the Maltsev condition $(\{\cdot\}, \{\text{group laws}\})$.
- **Example 2.** $\mathcal{V}$ has a *symmetric binary term* iff $\mathcal{V}$ satisfies the Maltsev condition $(\{\circ\}, \{x \circ y \approx y \circ x\})$.

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- **Example 3.** $\mathcal{V}$ has a *Maltsev term* iff $\mathcal{V}$ satisfies the Maltsev condition $(\{p\}, \{p(x,x,y) \approx y \approx p(y,x,x)\})$.

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- **Example 3.** $\mathcal{V}$ has a *Maltsev term* iff $\mathcal{V}$ satisfies the Maltsev condition $(\{p\}, \{p(x,x,y) \approx y \approx p(y,x,x)\})$.
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- **Example 5.** A *d-cube term* for $\mathcal{V}$ is a term $C$ witnessing that $\mathcal{V}$ satisfies the Maltsev condition $(\{C\}, \Sigma)$ where

$$\Sigma = \left\{ \text{rows of } C\left( \underbrace{\begin{bmatrix} x \\ y \\ \vdots \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ \vdots \\ y \end{bmatrix}, \ldots, \begin{bmatrix} y \\ y \\ \vdots \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ \vdots \\ y \end{bmatrix}, \ldots}_{d\text{-tuples in } x, y, \text{ with at least one } x} \right) \approx \begin{bmatrix} y \\ y \\ \vdots \\ y \end{bmatrix} \right\}.$$

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- Examples 2–5 are *linear* Maltsev conditions; Example 1 is not.

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## Theorem 2 (Peter Mayr – ASz)

*If* $\mathbf{A}$ *is a finite algebra (in a finite language) such that*

$(*)$ *for every SI in* $\mathbb{HS}(\mathbf{A})$ *with abelian monolith* $\mu$*,* $(0 : \mu)$ *is supernilpotent,*

*then* $\mathrm{SMP}(\mathcal{K}) \in \mathsf{P}$.

# Existence of a Cube Term

- The following conditions on a finite algebra $\mathbf{A}$ are equivalent:
  - for some polynomial $p$, $|\mathrm{Sub}(\mathbf{A}^m)| \leq 2^{p(m)}$;
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  - There exists a 10-element band $\mathbf{S}$ and a 9-element quotient $\mathbf{S}/\theta$ such that $\mathrm{SMP}(\mathbf{S}) \in \mathsf{P}$ and $\mathrm{SMP}(\mathbf{S}/\theta)$ is NP-complete. [Steindl, 2017/8]

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- The query on $h$ is *improper* if $h$ does not encode a concept in $\Gamma$.

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## Theorem 1 (Jeff Shriner)

*There exists a finite algebra $\mathbf{A}_\mathcal{M}$ (in the language $\mathcal{F} \cup \mathcal{H}$) such that $\mathbf{A}_\mathcal{M}$ satisfies $\mathcal{M}$, and $\mathrm{SMP}(\mathbf{A}_\mathcal{M})$ is at least as hard as $\mathrm{SMP}(\mathbf{A})$.*

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*There exists a finite algebra $\mathbf{A}_\mathcal{M}$ (in the language $\mathcal{F} \cup \mathcal{H}$) such that $\mathbf{A}_\mathcal{M}$ satisfies $\mathcal{M}$, and* $\mathrm{SMP}(\mathbf{A}_\mathcal{M})$ *is at least as hard as* $\mathrm{SMP}(\mathbf{A})$.

- For Kozik's algebra $\mathbf{K}$,
  $\mathbf{K}_\mathcal{M}$ satisfies $\mathcal{M}$ and $\mathrm{SMP}(\mathbf{K}_\mathcal{M})$ is EXPTIME-complete.

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Let $\mathbf{A}$ be any finite algebra (in a finite language $\mathcal{F}$).
Let $\mathcal{M} = (\mathcal{H}, \Sigma)$ be a consistent, strong linear Maltsev condition such that $\Sigma$ does not entail cube identities for any $h \in \mathcal{H}$.
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**Step 1:** Definition of $\mathbf{A}_\mathcal{M}$; checking that $\mathbf{A}_\mathcal{M}$ satisfies $\mathcal{M}$.

**Step 2:** Polynomial time reduction of $\mathrm{SMP}(\mathbf{A})$ to $\mathrm{SMP}(\mathbf{A}_\mathcal{M})$. □

# Step 1: Constructing $\mathbf{A}_{\mathcal{M}}$

Given: $\mathbf{A} = (A; \mathcal{F})$ and $\mathcal{M} = (\mathcal{H}, \Sigma)$.

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$\mathbf{A}_{\mathcal{M}}$ satisfies the Maltsev condition $\mathcal{M}$.

# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$)

Input for SMP($\mathbf{A}$):
$a_1, \ldots, a_n, b \in \mathbf{A}^m$

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|---|---|---|---|
| | $a_1, \dots, a_n, b \in \mathbf{A}^m$ | $\overset{\text{const time}}{\longmapsto}$ | $a_1, \dots, a_n, b \in \mathbf{A}_{\mathcal{M}}^m$ |
| **Goal:** | $b \in \langle a_1 \dots, a_n \rangle_{\mathbf{A}^m}$ | $\iff$ | $b \in \langle a_1, \dots, a_n \rangle_{\mathbf{A}_{\mathcal{M}}^m}$ |

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| | | $\overset{!!!}{\Longleftarrow}$ | |

Need to show: If $b = t(a_1 \dots, a_n)$ for some term $t$ in the language $\mathcal{F} \cup \mathcal{H}$, then $b = t'(a_1 \dots, a_n)$ for some term $t'$ in the language $\mathcal{F}$.

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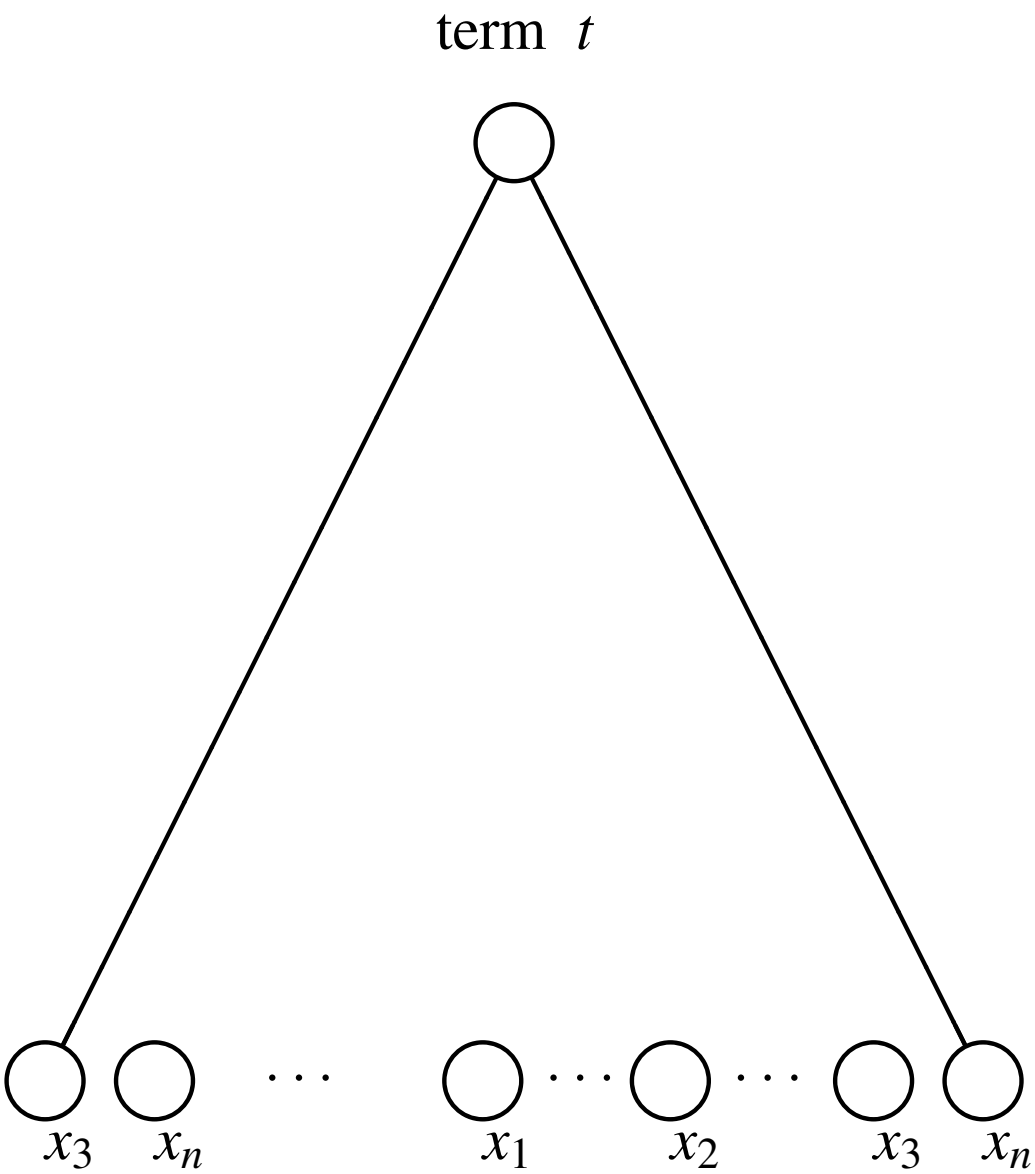
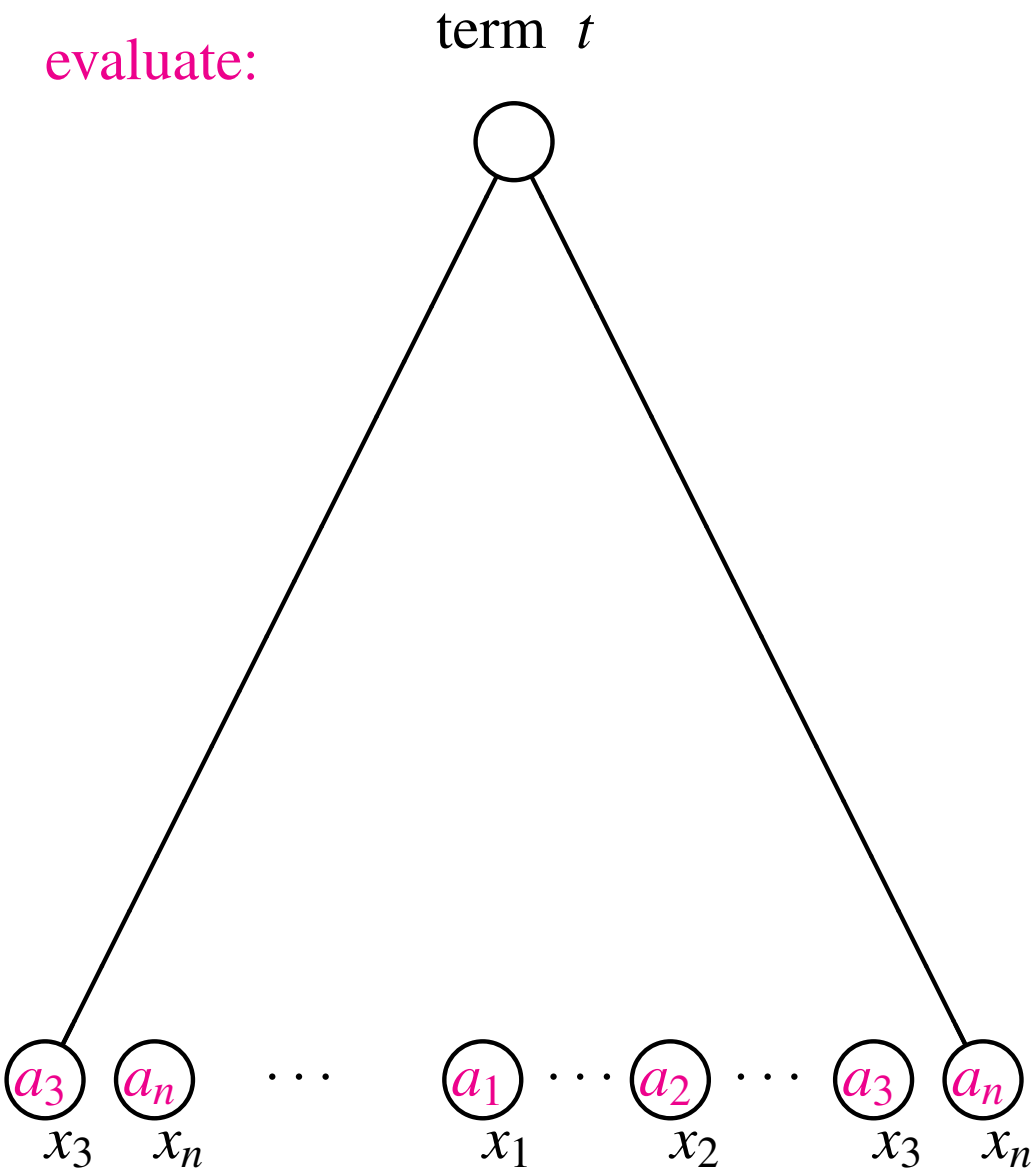
Enough: to eliminate one occurrence of a symbol $h \in \mathcal{H}$ from $t$ at a time.

# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)



# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)

evaluate:



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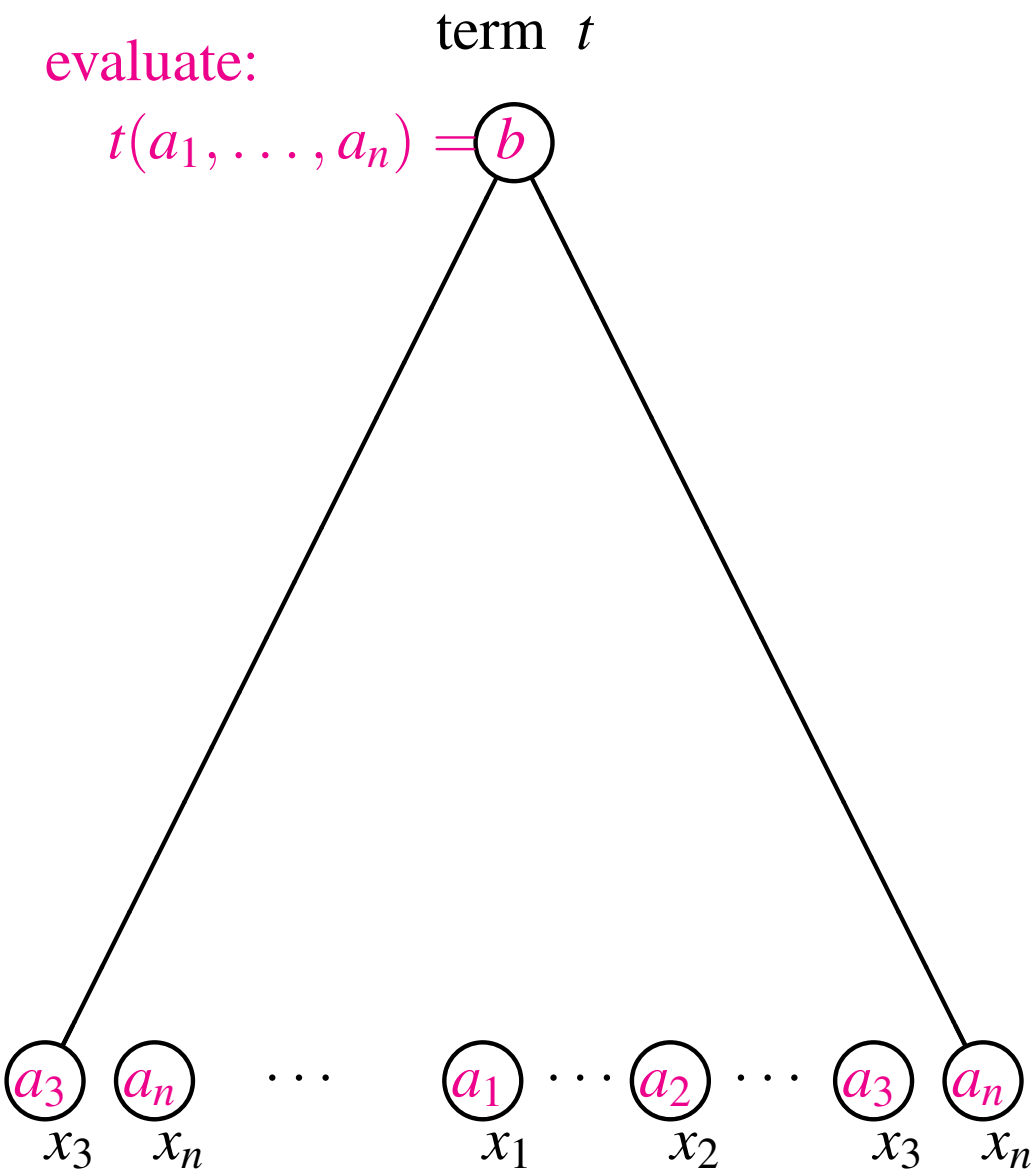

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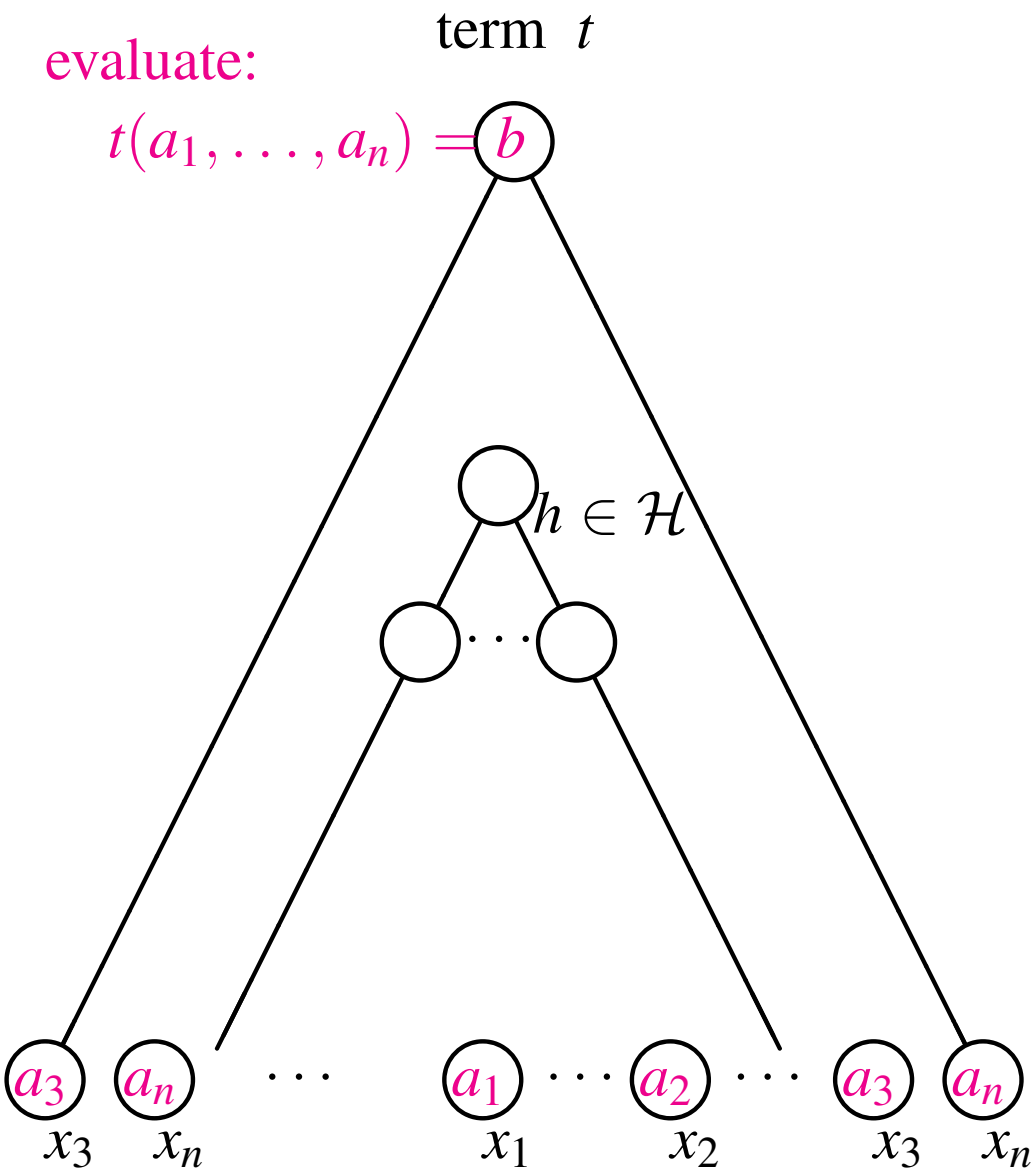

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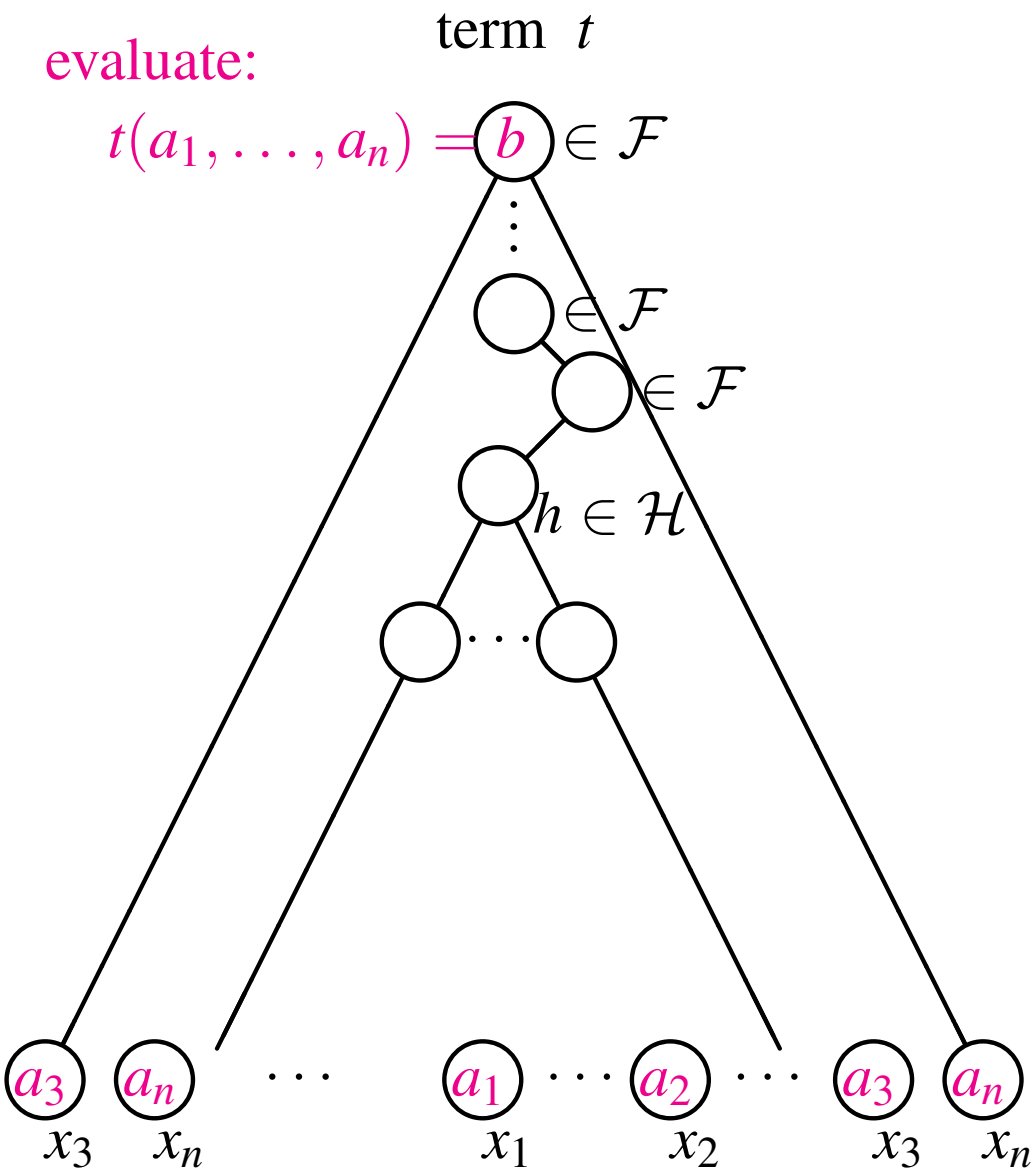

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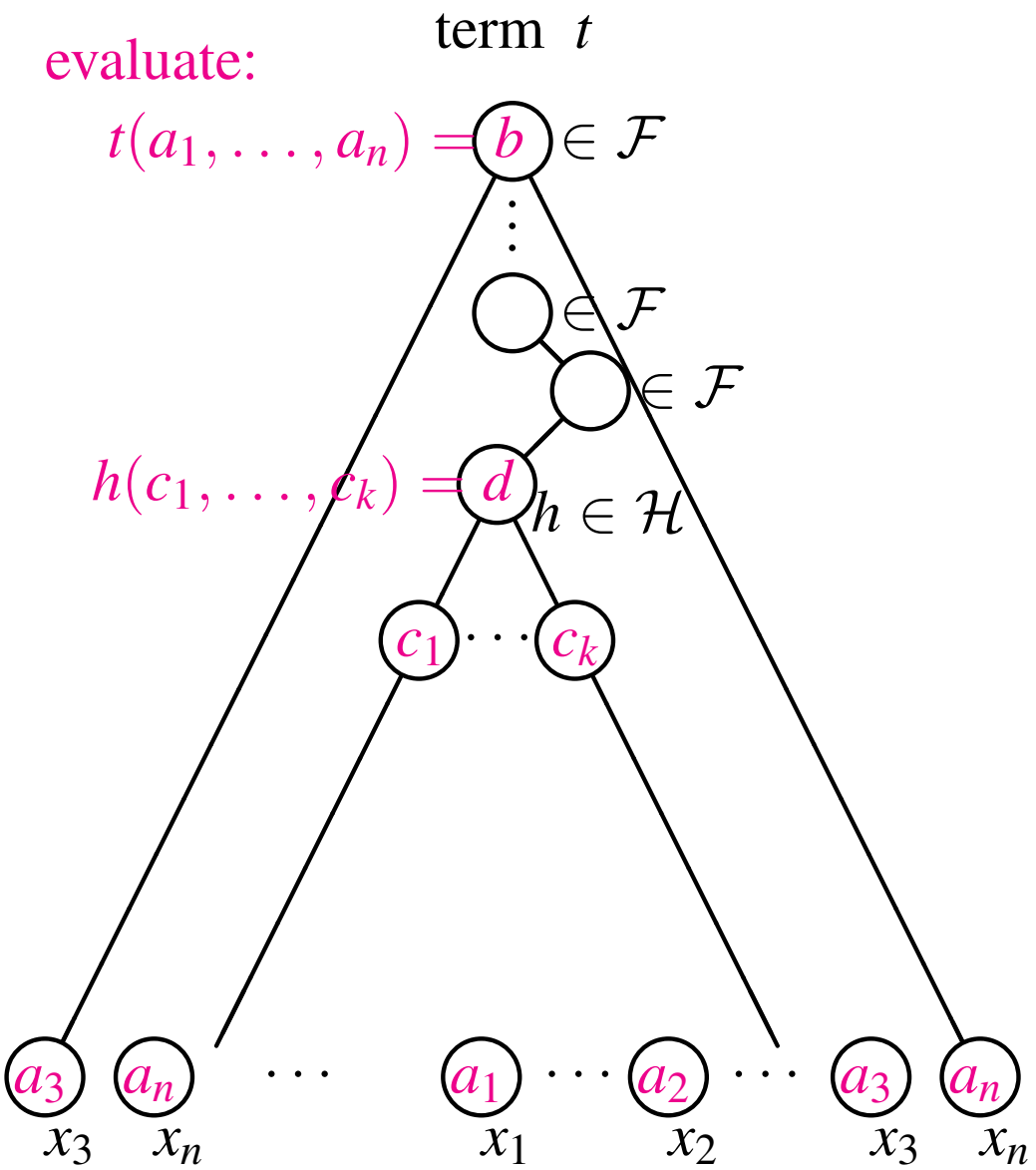

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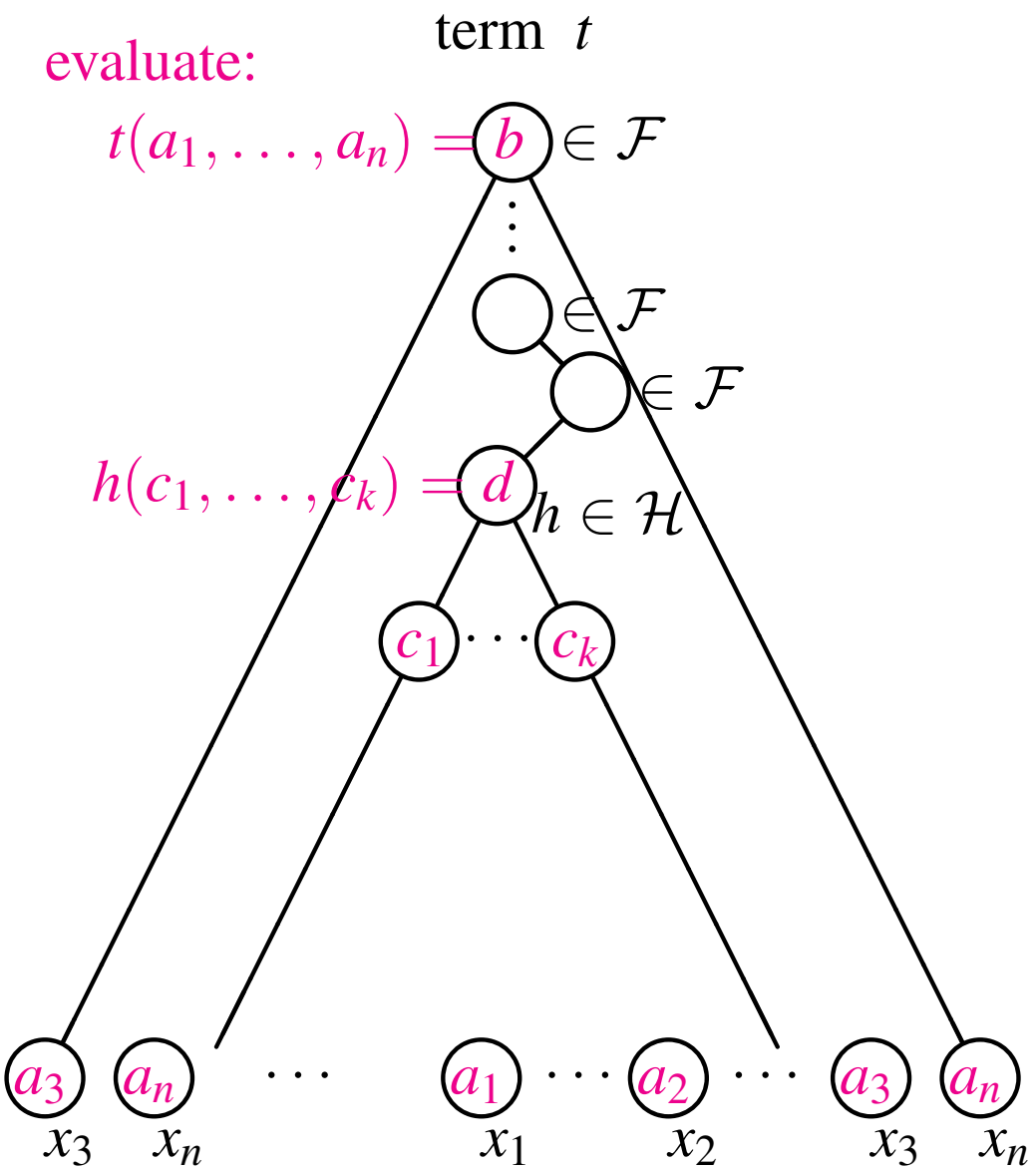


$d$ has no 0 coord's

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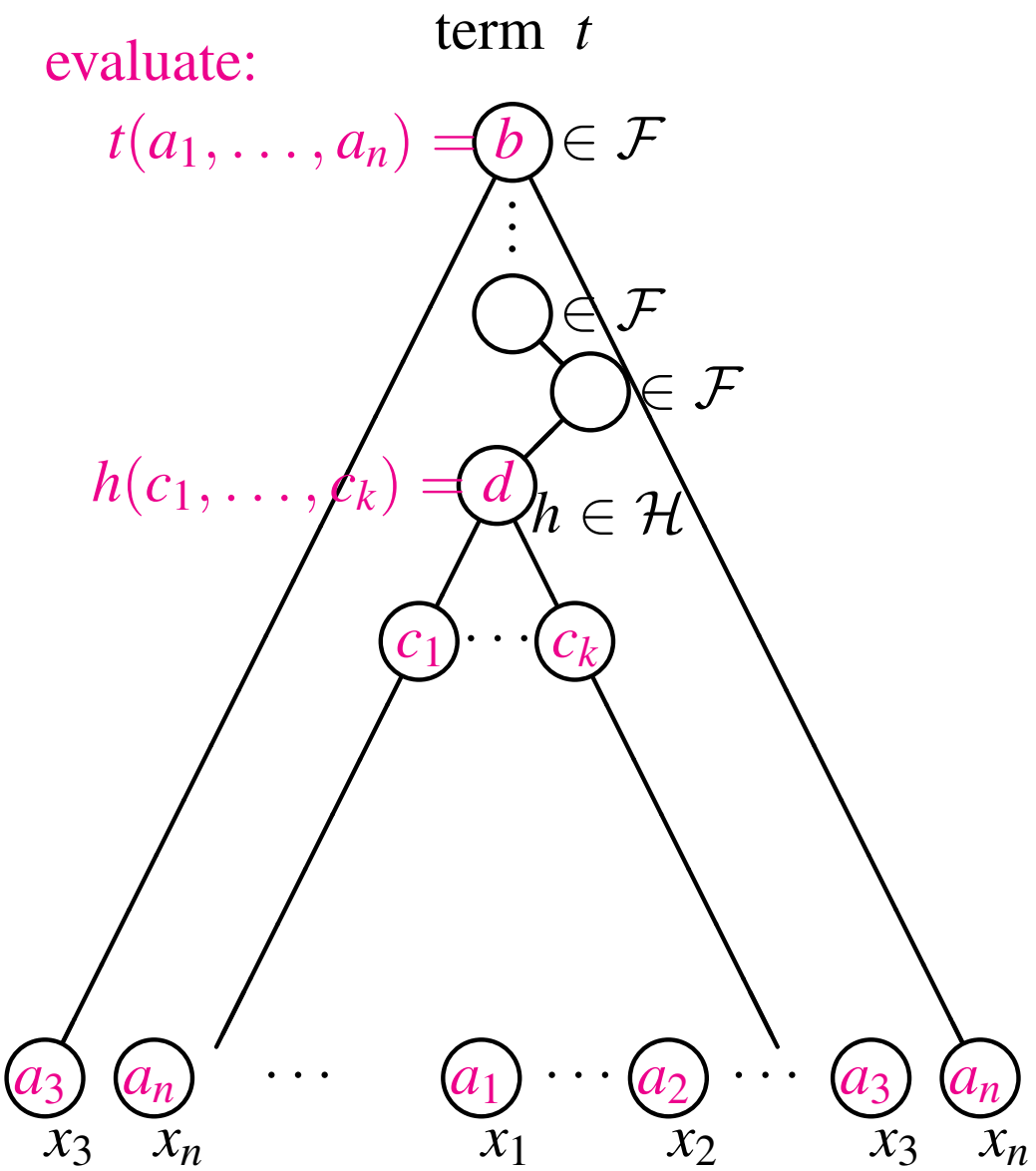


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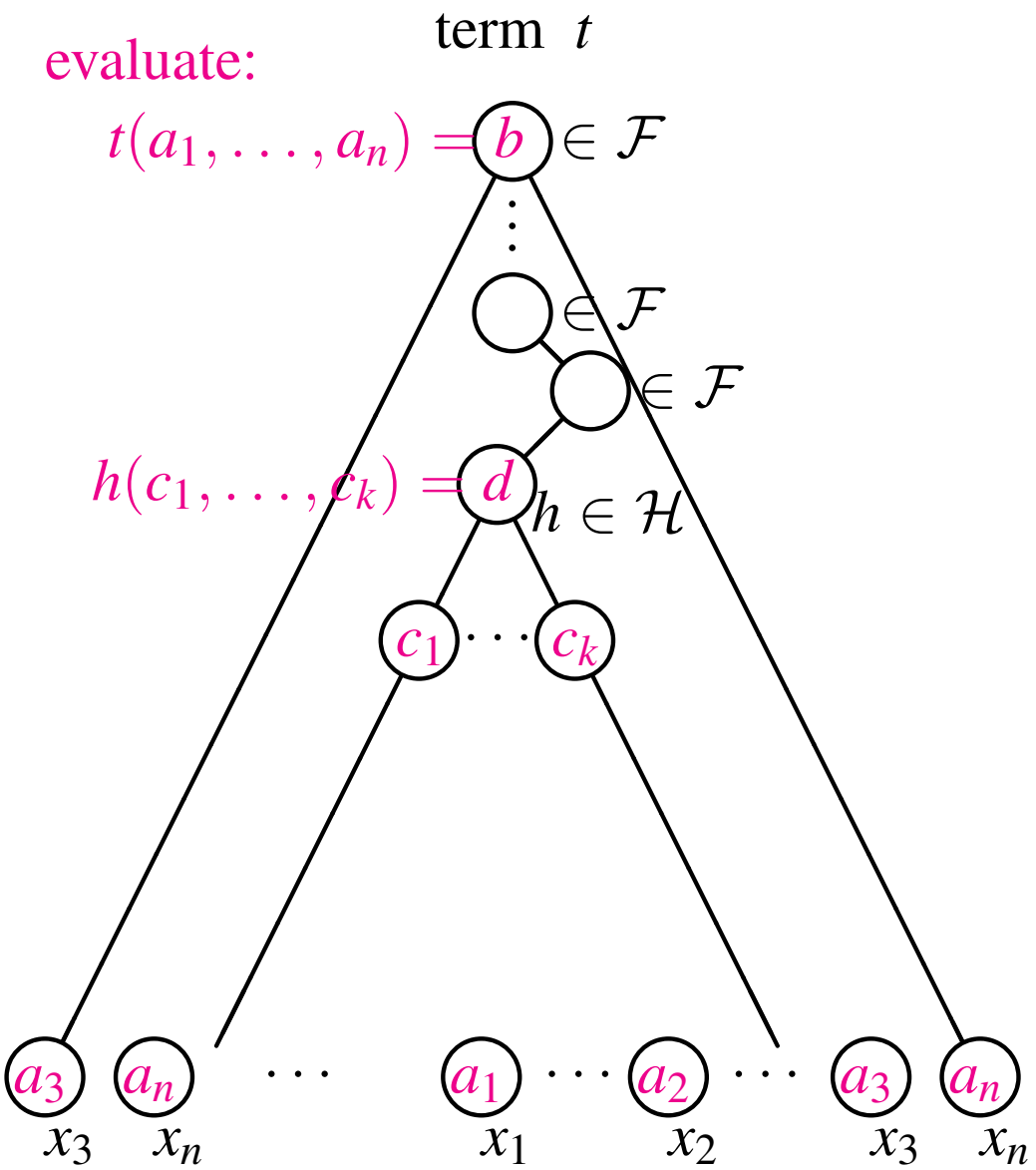
**Case 1:** $d = c_i$ for some $i$

# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)

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Eliminate $h$.

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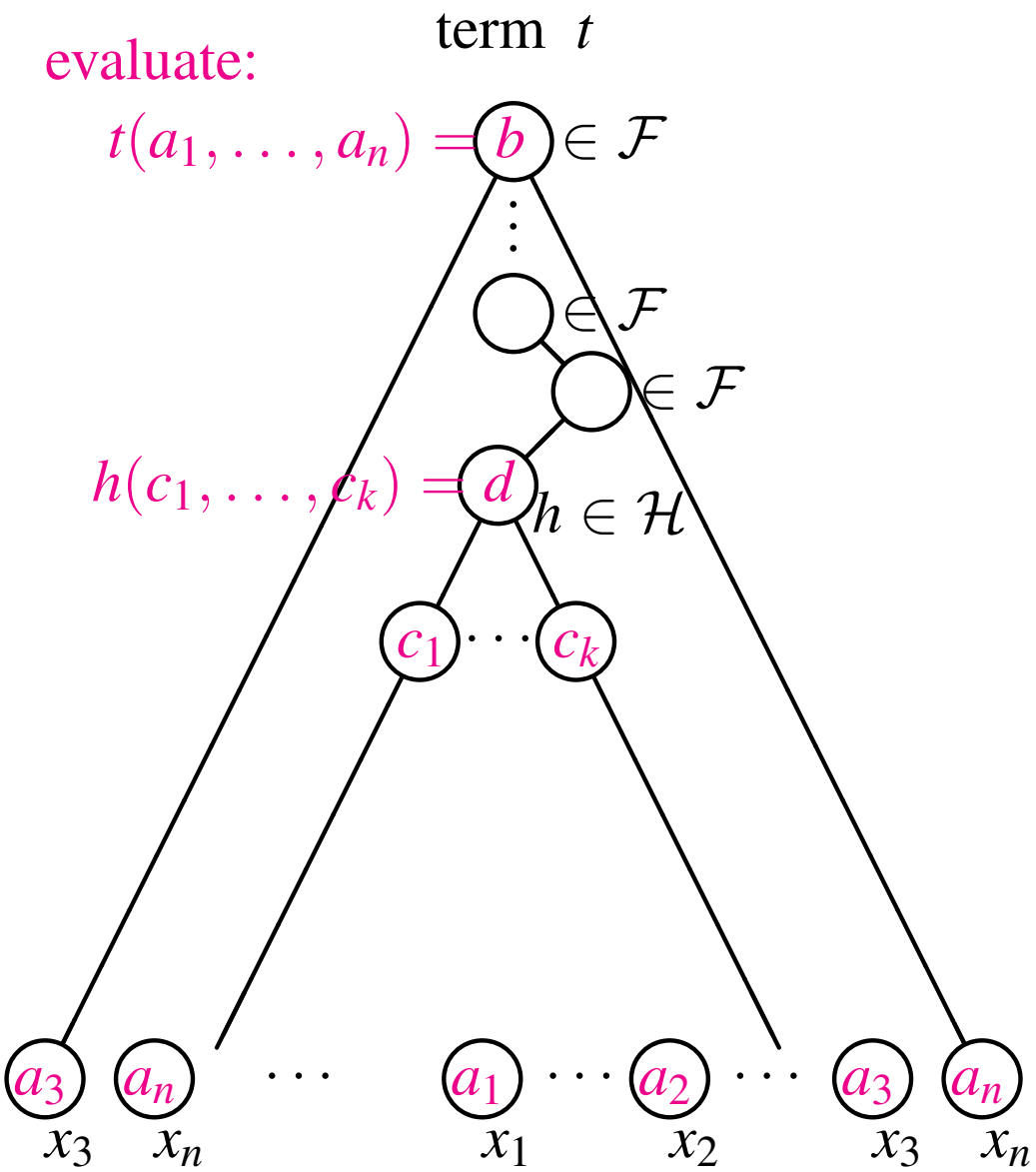


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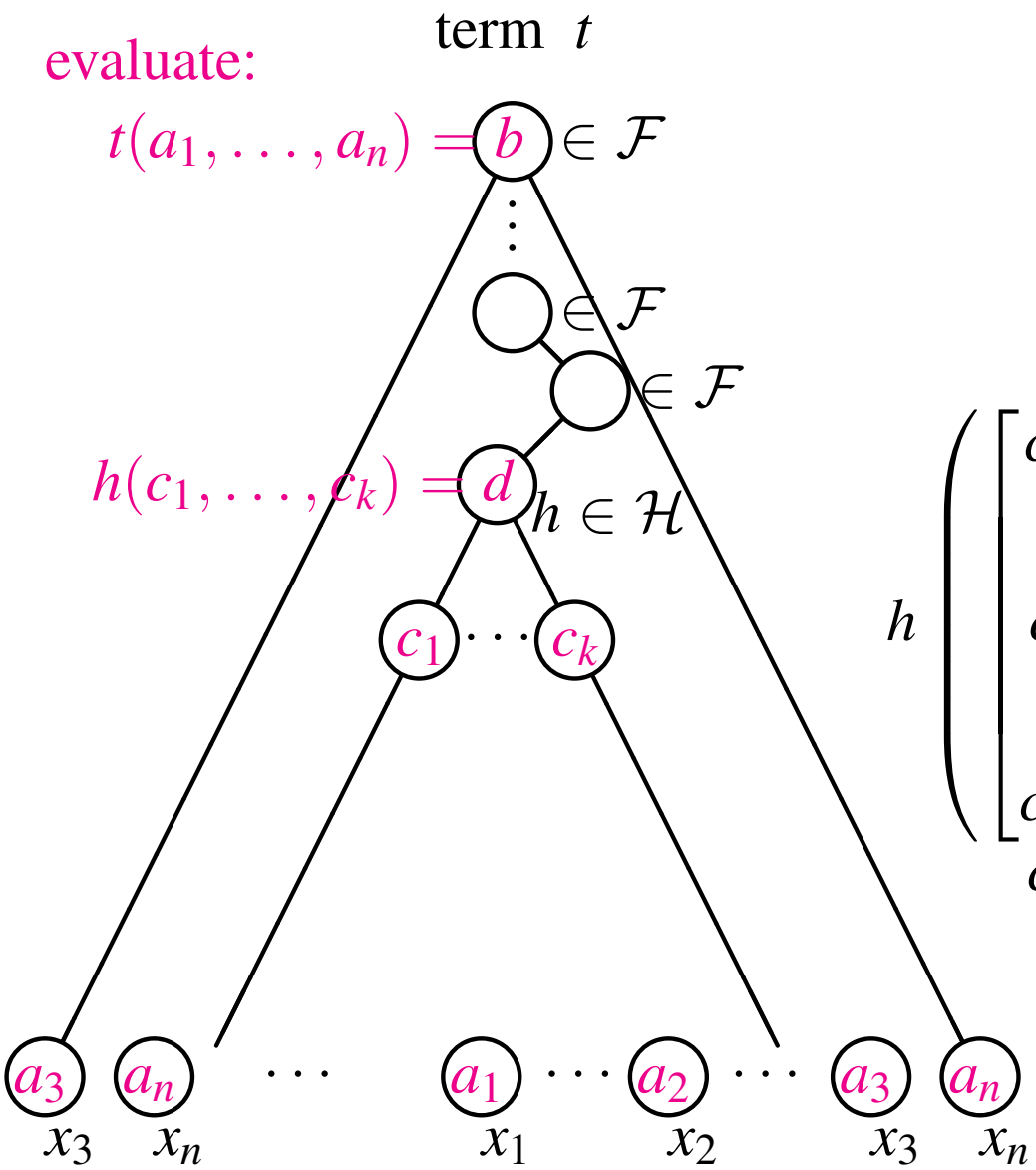


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**Case 2:** $d \neq c_i$ for every $i$

$$h\left(\underset{c_1}{\begin{bmatrix} c_{11} \\ \vdots \\ c_{j1} \\ \vdots \\ c_{m1} \end{bmatrix}} \cdots \underset{c_i}{\begin{bmatrix} c_{1i} \\ \vdots \\ c_{ji} \\ \vdots \\ c_{mi} \end{bmatrix}} \cdots \underset{c_k}{\begin{bmatrix} c_{1k} \\ \vdots \\ c_{jk} \\ \vdots \\ c_{mk} \end{bmatrix}}\right) = \underset{d}{\begin{bmatrix} d_1 \\ \vdots \\ d_j \\ \vdots \\ d_m \end{bmatrix}}$$

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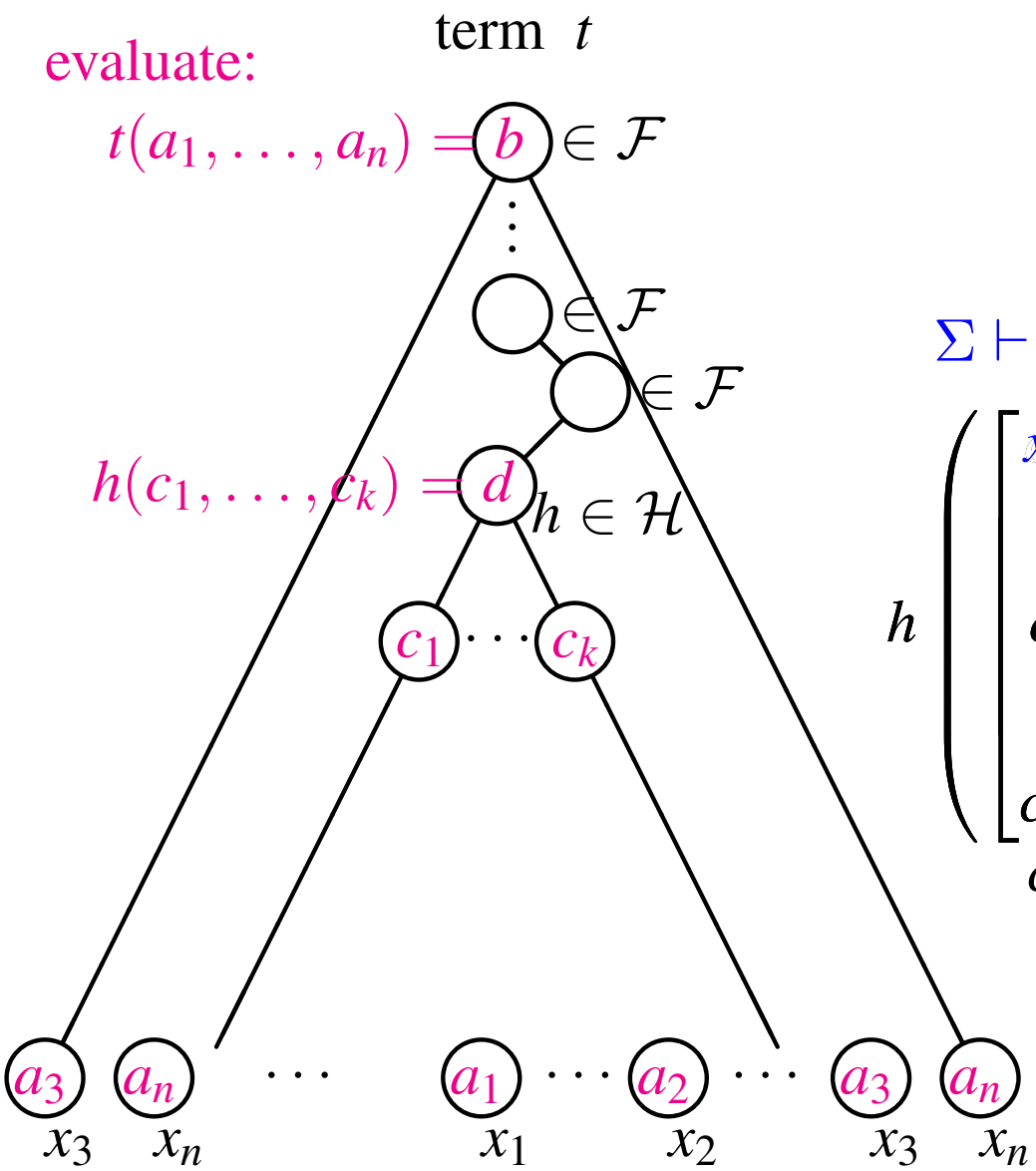


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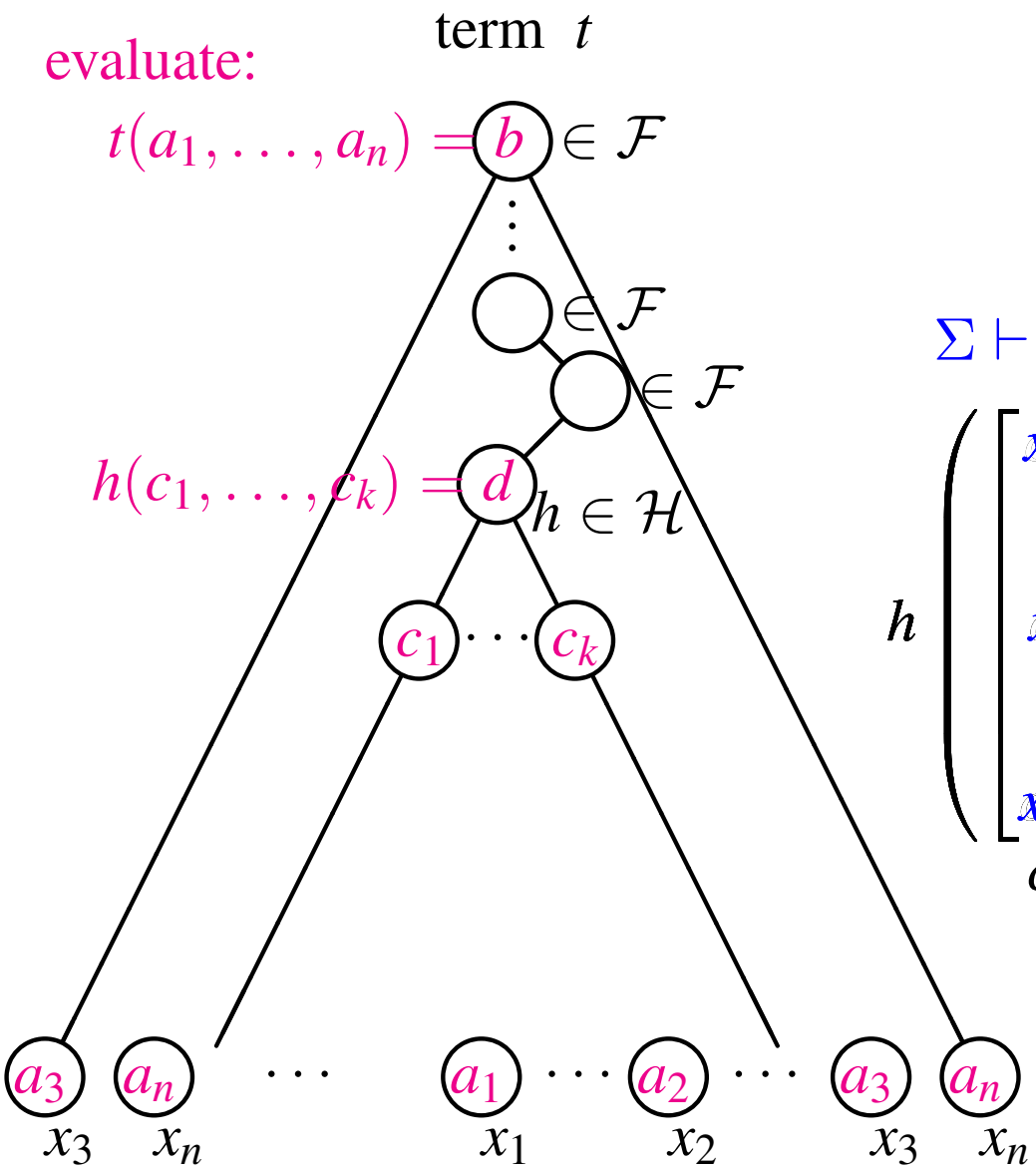
**Case 2:** $d \neq c_i$ for every $i$

$$\Sigma \vdash h\left(\begin{bmatrix} x_{11} \\ \vdots \\ c_{j1} \\ \vdots \\ c_{m1} \end{bmatrix}_{c_1} \cdots \begin{bmatrix} x_{1i} \\ \vdots \\ c_{ji} \\ \vdots \\ c_{mi} \end{bmatrix}_{c_i} \cdots \begin{bmatrix} x_{1k} \\ \vdots \\ c_{jk} \\ \vdots \\ c_{mk} \end{bmatrix}_{c_k}\right) = \begin{bmatrix} y \\ \vdots \\ d_j \\ \vdots \\ d_m \end{bmatrix}_{d}$$

# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)

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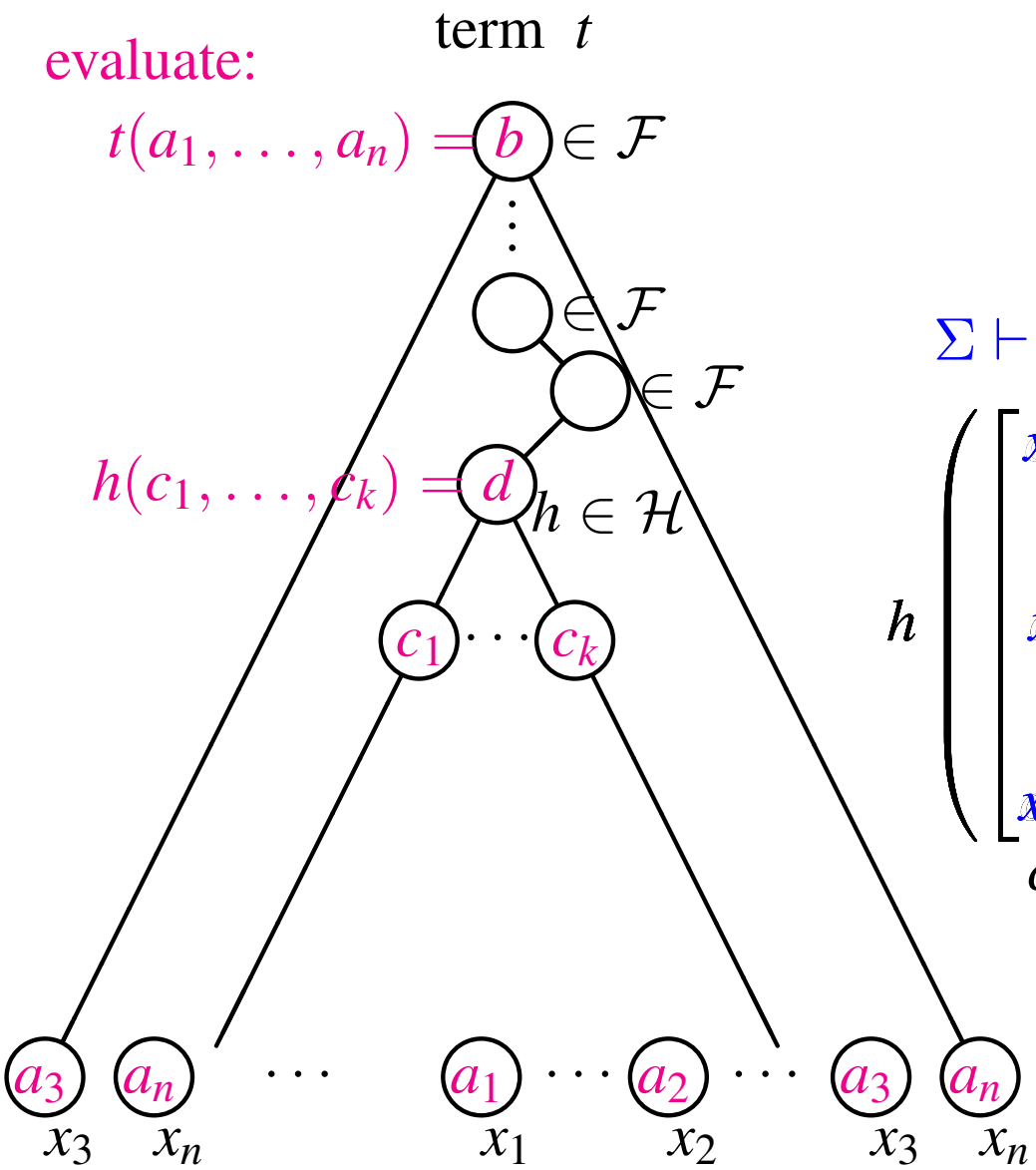
# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)

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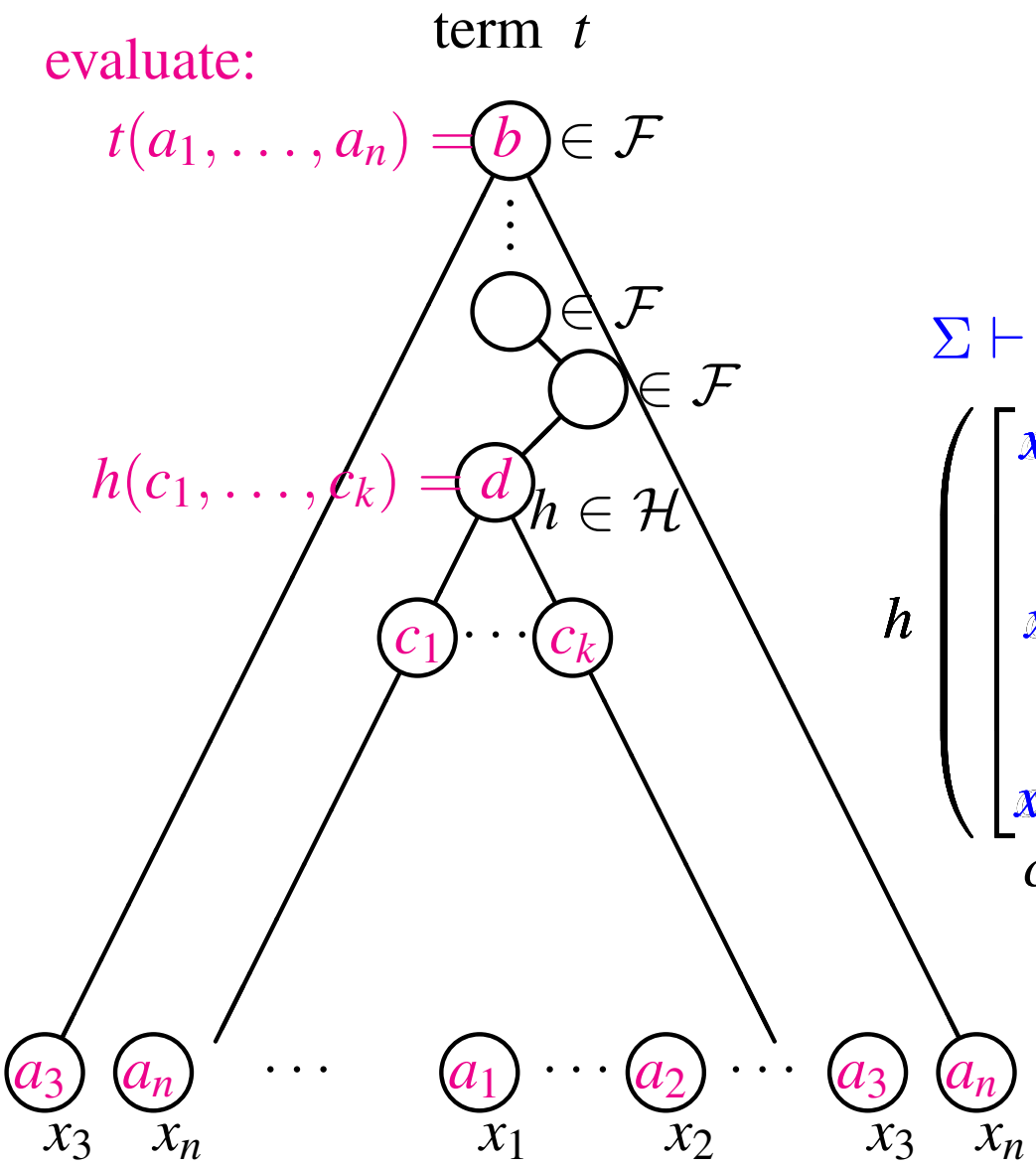
$c_1 \quad c_i \quad c_k \quad d$

Replace each $x_{qr} \neq y$ by $x$.

# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)

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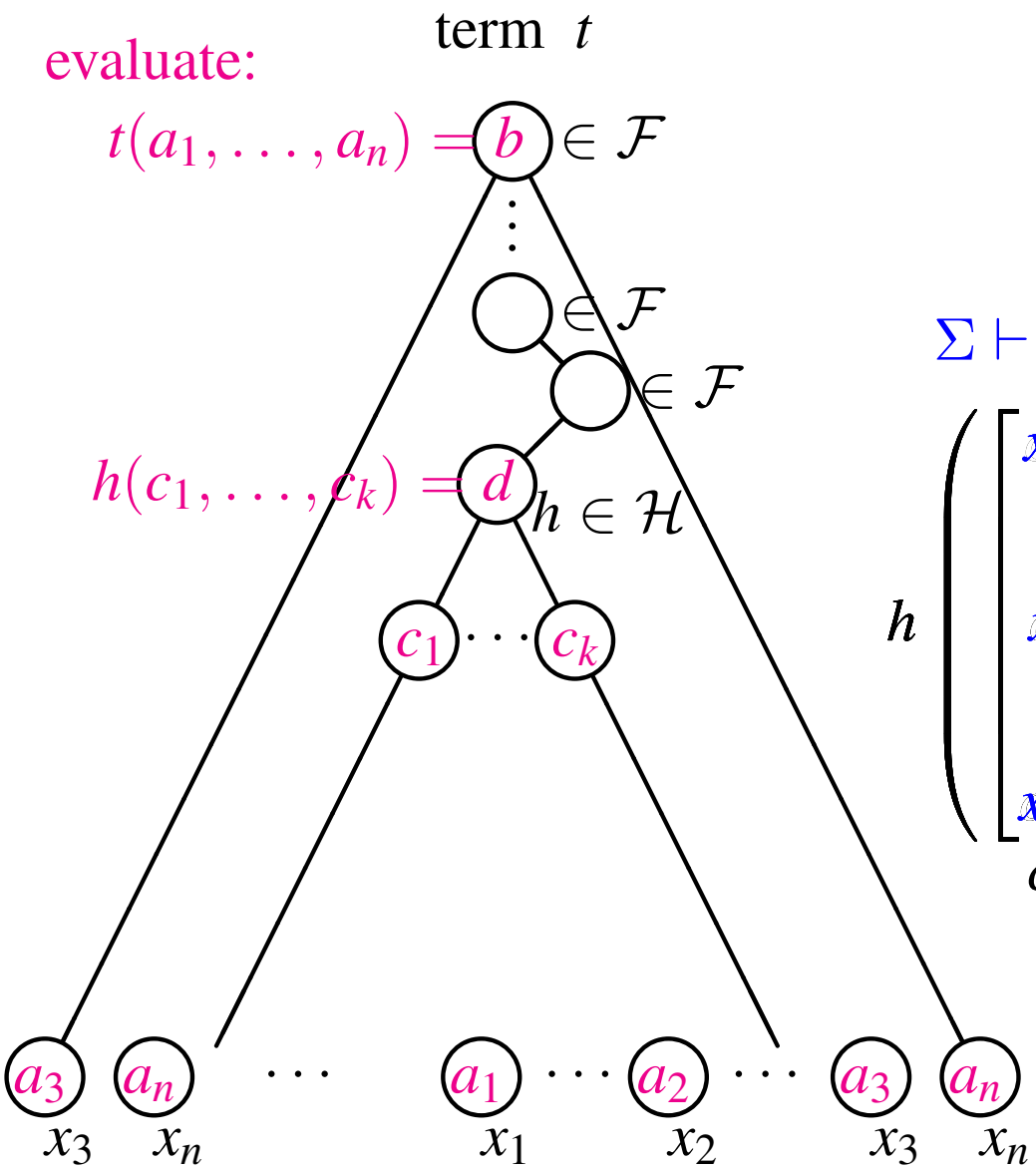
$c_1 \quad c_i \quad c_k \quad d$

Replace each $x_{qr} \neq y$ by $x$.
We get cube identities for $h$.

# Step 2: P-time reduction of SMP($\mathbf{A}$) to SMP($\mathbf{A}_{\mathcal{M}}$) (Cont'd)

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$c_1$ $c_i$ $c_k$ $d$

Replace each $x_{qr} \neq y$ by $x$.
We get cube identities for $h$.
Impossible!

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Throughout:

$\mathcal{V}$: a variety in a finite language, $\mathcal{K}$: a finite set of finite algebras in $\mathcal{V}$

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Extends the earlier result [Bulatov–Mayr–ASz, 201?] that

- same conclusion is true under the stronger assumption that

  $(**)$ for every SI in $\mathbb{HS}\,\mathcal{K}$ with abelian monolith $\mu$, $(0 : \mu)$ is abelian (equivalently: $\mathcal{V}(\mathcal{K})$ is residually small).

# Idea of Proof

Proof in the abelian case relies on:

(1) $\mathrm{SMP}(\mathcal{K}) \in \mathsf{P} \Leftrightarrow \mathrm{SMP}(\mathbb{HS}\,\mathcal{K}) \in \mathsf{P} \Leftrightarrow \mathrm{SMP}(\prod_{\mathbf{A}\in\mathbb{HS}\,\mathcal{K}} \mathbf{A}) \in \mathsf{P}$; may assume $\mathcal{K} = \mathbb{HS}\,\mathcal{K}$.

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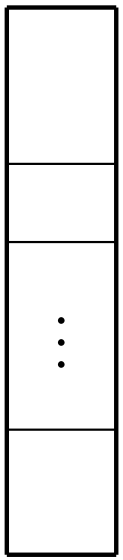
- $\mathrm{SMP}(\mathbf{M}) \in \mathsf{P}$ if $\mathbf{M}$ is a nilpotent Maltsev algebra of prime power order in a finite language. [Mayr, 2012]

# The Reduction: Algebras from Congruences

$$\mathbf{A}, \alpha$$
$$(|\mathbf{A}/\alpha| = n)$$



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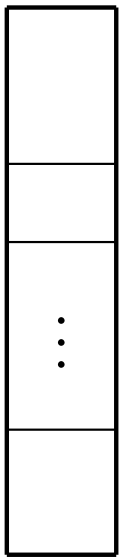
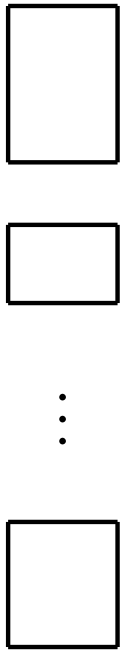


multisorted alg
$\mathrm{MS}(\mathbf{A}, \alpha)$



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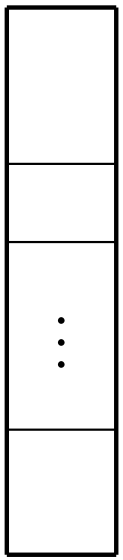

$f$

(say, binary)

operations

multisorted alg

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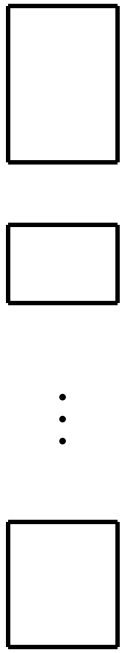

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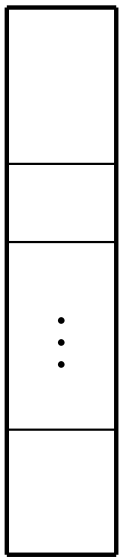

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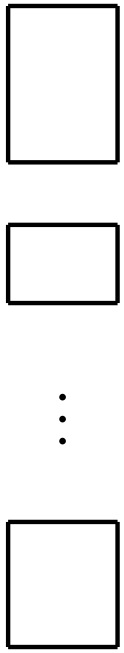

$f_{B,C;D}$
($B$, $C$, $D$ $\alpha$-classes,
$D \supseteq f[B, C]$)

operations

# The Reduction: Algebras from Congruences

operations

$\mathbf{A}, \alpha$
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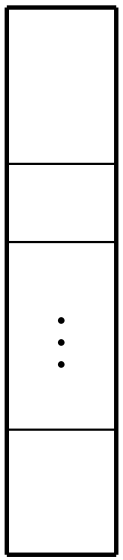

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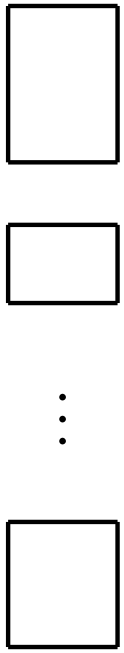
multisorted alg
$\mathrm{MS}(\mathbf{A}, \alpha)$



$f_{B,C;D}$
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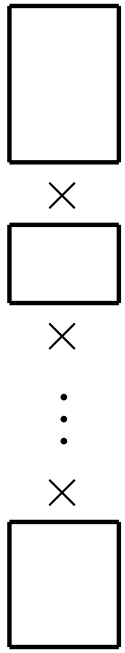
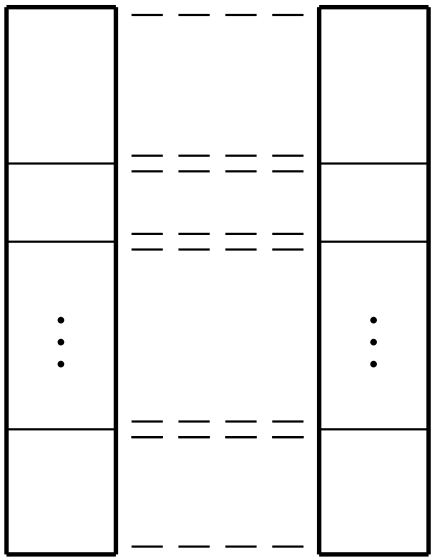
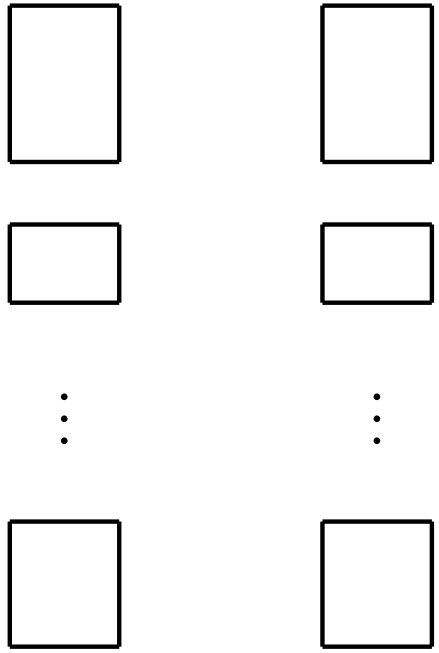
homogenization
$\mathrm{OS}(\mathbf{A}, \alpha)$



$d$ ($n$-ary diag op),
$\hat{f}_{B,C;D}$
$=$ (proj's, $f_{B,C;D}$, proj's)
$\uparrow_D$

# Subpowers





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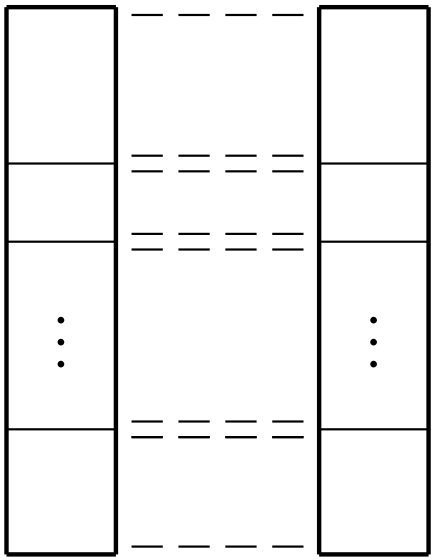
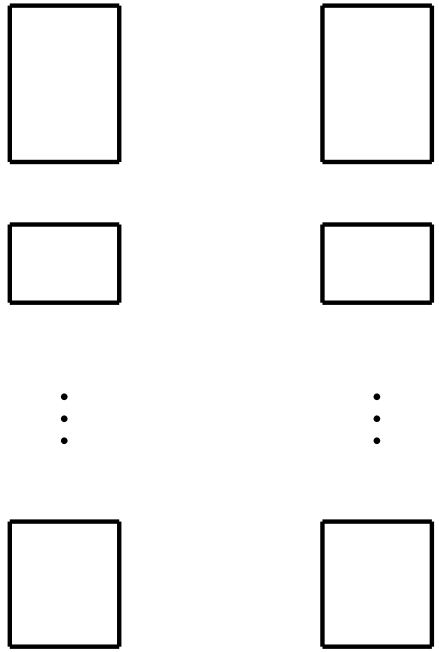


$\mathbf{A}^m[\alpha]$



$\mathbf{MS}(\mathbf{A}, \alpha)^m$

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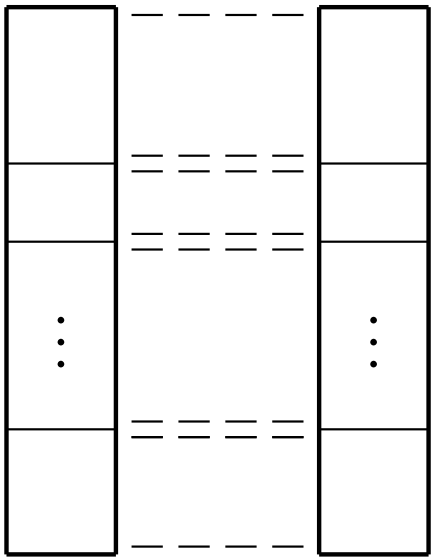

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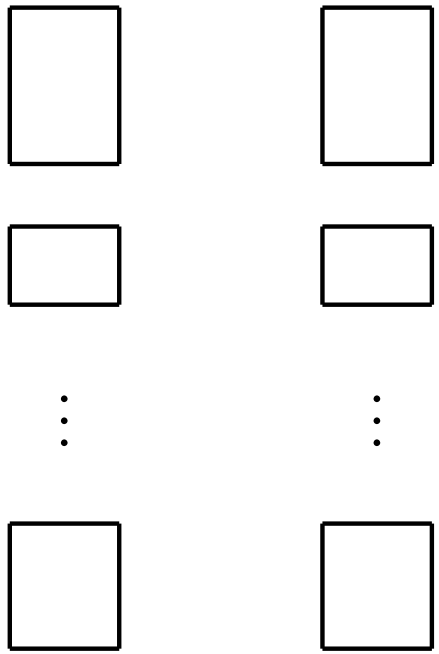

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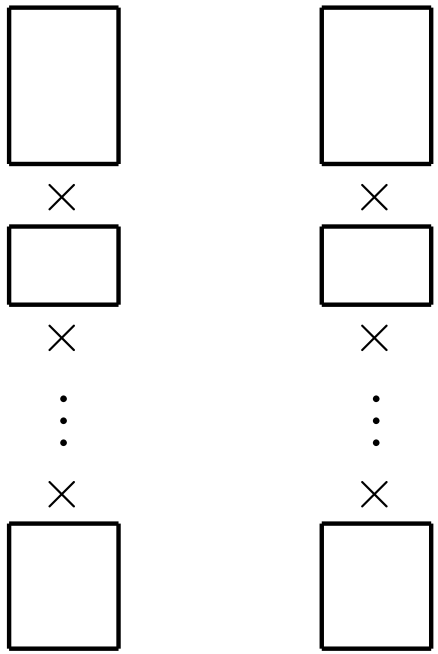

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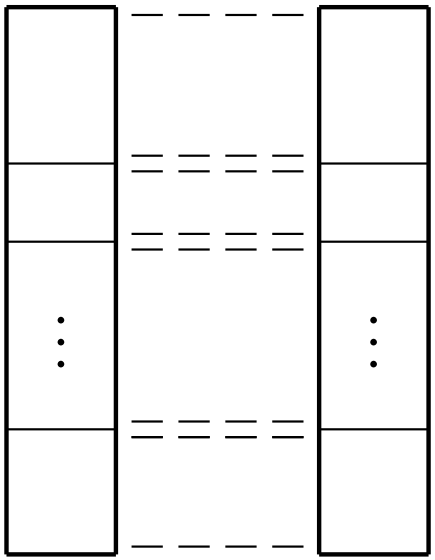

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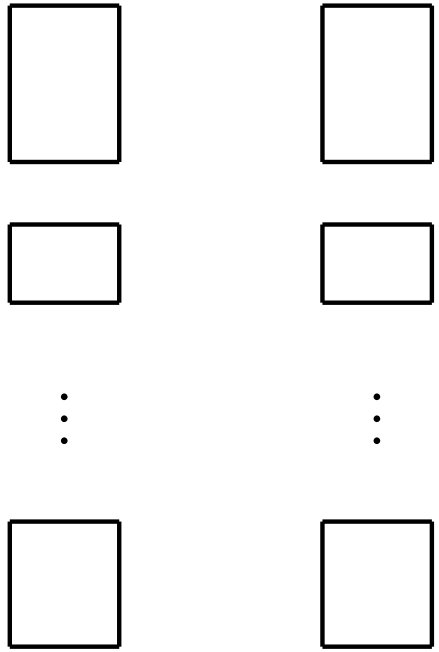

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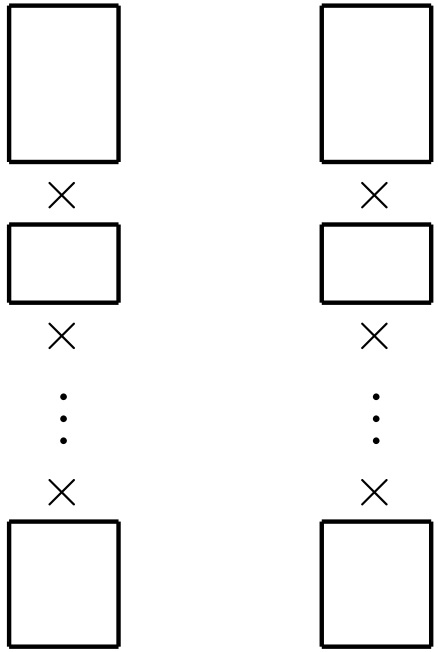

$\mathrm{OS}(\mathbf{A}, \alpha)^m$

subalgebras

$\mathbf{B} \longleftrightarrow \mathrm{MS}(\mathbf{B}, \alpha^m)$

($\mathbf{B}$ split into $n$ $\alpha^m$-classes)

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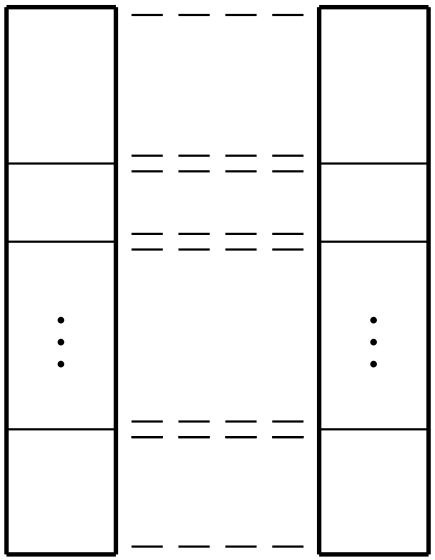

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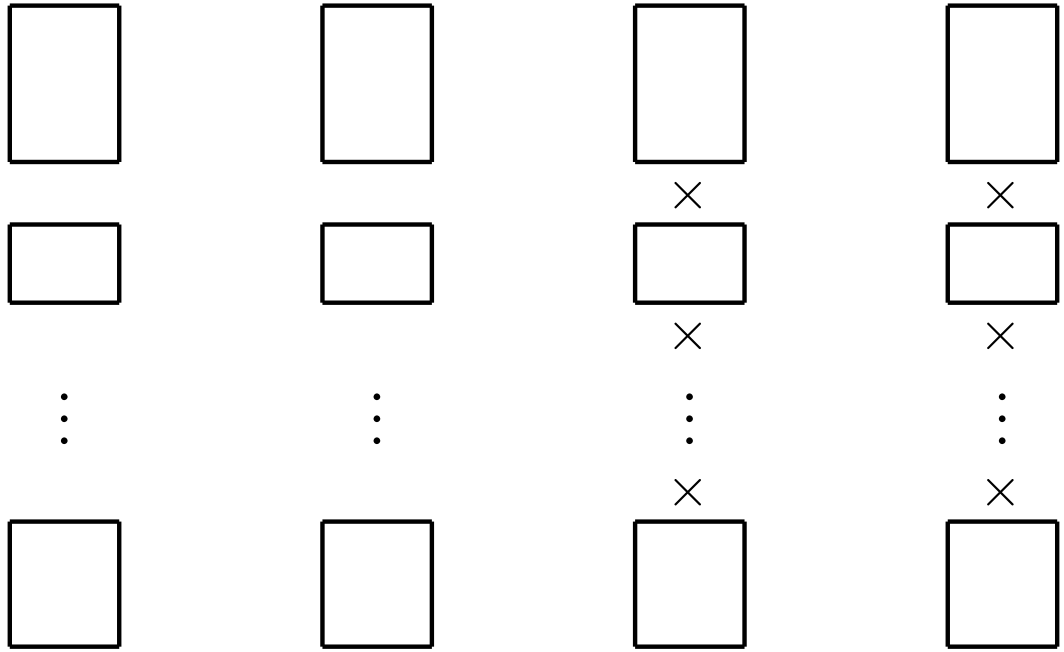

$\mathrm{MS}(\mathbf{A},\alpha)^m$



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subalgebras

$\mathbf{B}$ $\longleftrightarrow$ $\mathrm{MS}(\mathbf{B},\alpha^m)$ $\xleftarrow{\text{if } \mathbf{B} \text{ meets all } n \ \alpha^m\text{-classes}}$ $\mathrm{OS}(\mathbf{B},\alpha^m)$

($\mathbf{B}$ split into $n$ $\alpha^m$-classes)

# Congruences

$$\mathbf{A}^m[\alpha] \qquad \mathbf{MS}(\mathbf{A},\alpha)^m \qquad \mathbf{OS}(\mathbf{A},\alpha)^m$$

$$\mathbf{B} \longleftrightarrow \mathbf{MS}(\mathbf{B},\alpha^m) \xleftrightarrow[\alpha^m\text{-classes}]{\text{if } \mathbf{B} \text{ meets all } n} \mathbf{OS}(\mathbf{B},\alpha^m)$$

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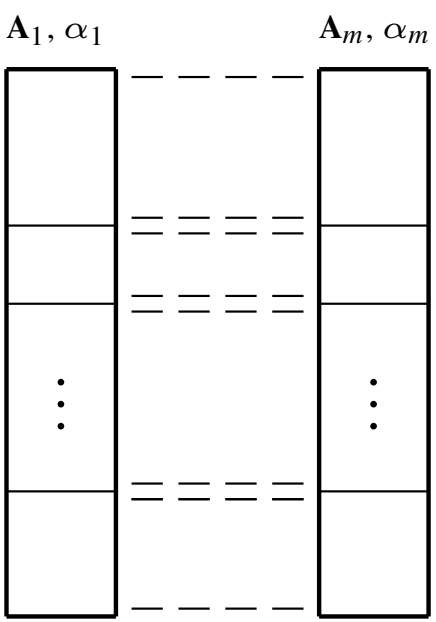
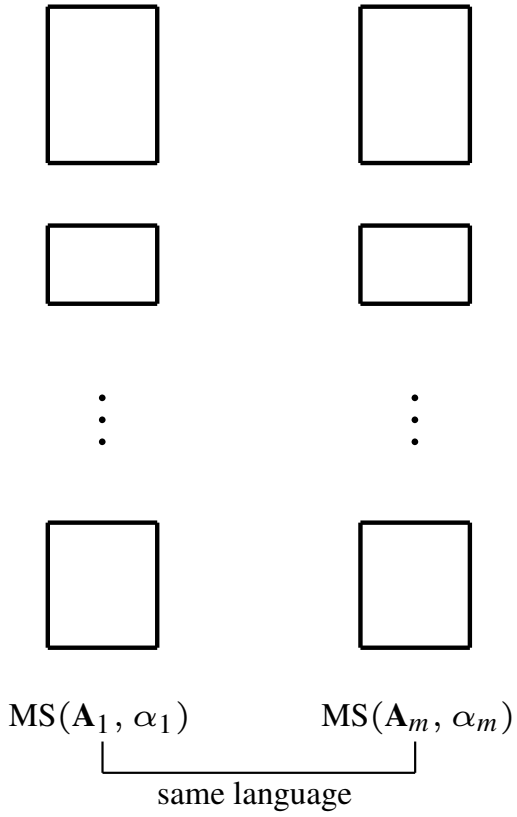



$\{(a_1, \ldots, a_m) : a_i \, \alpha_i f_i(a_1)\}$



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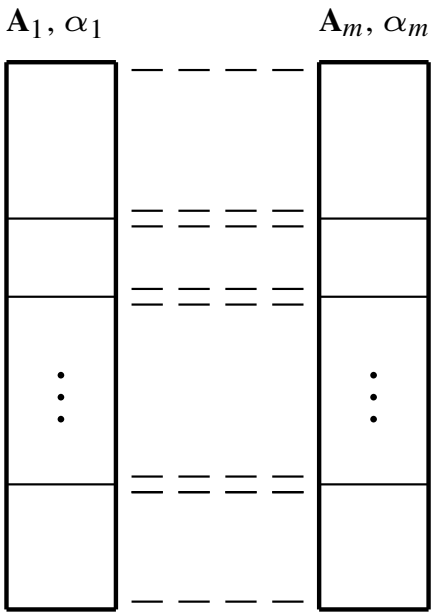


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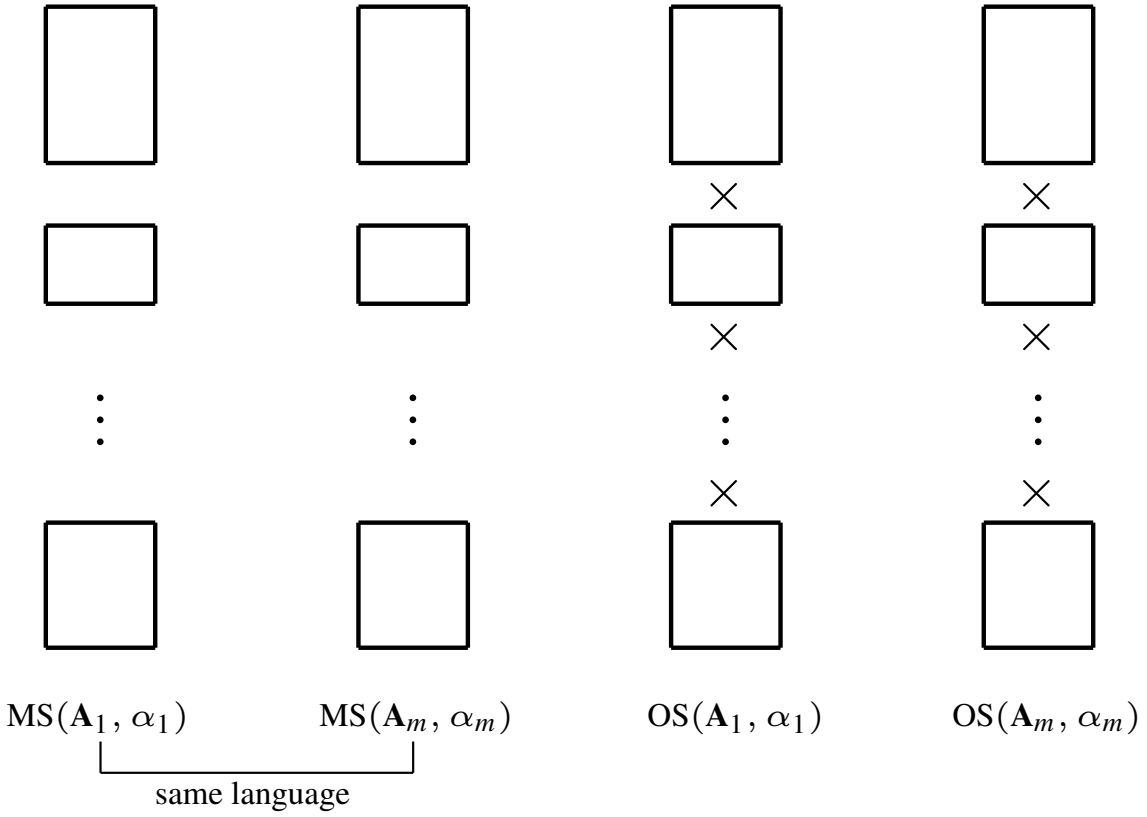

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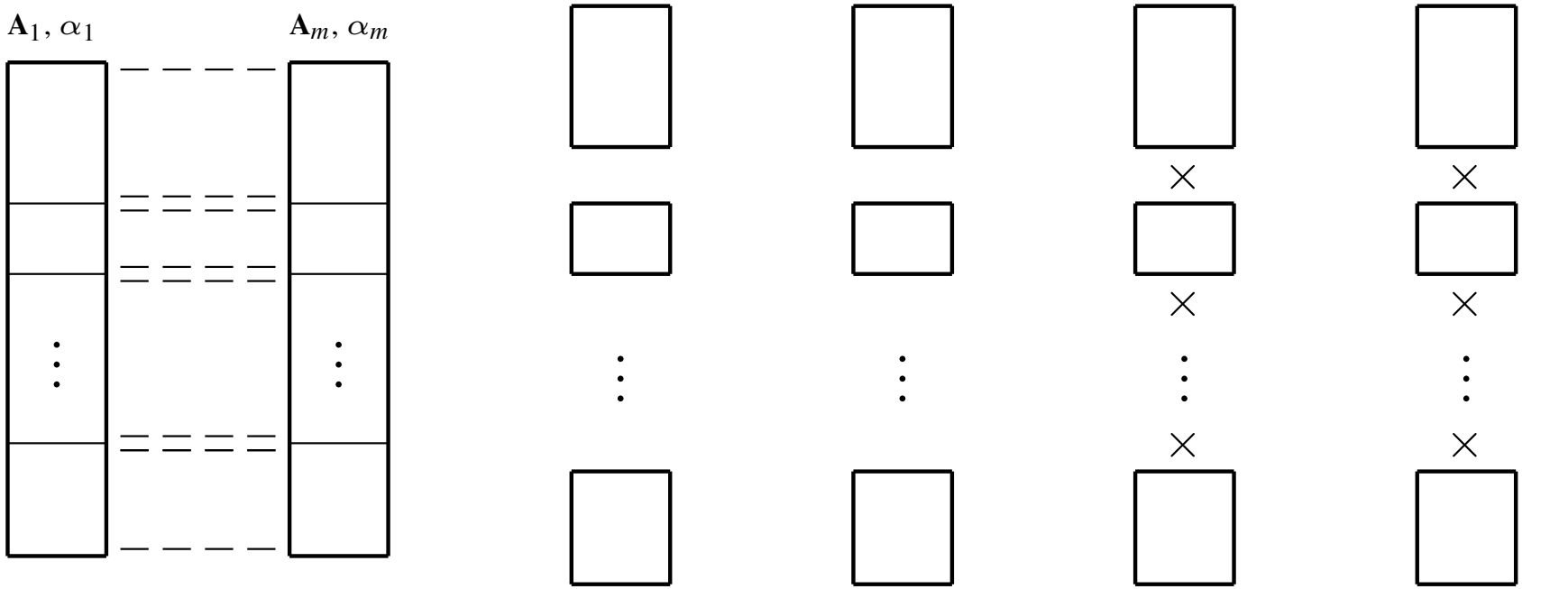


subalgs

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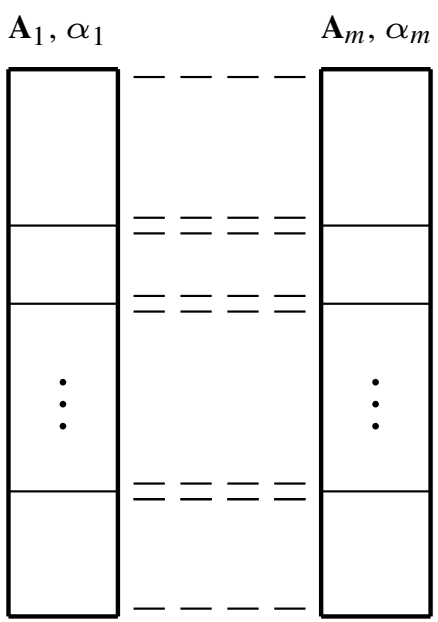


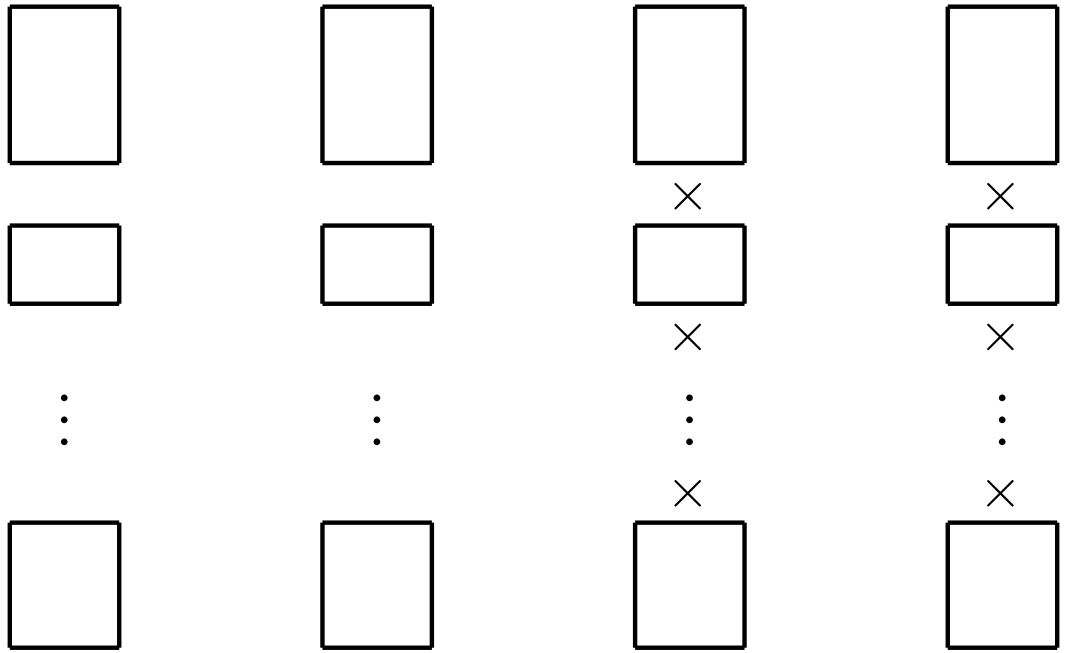

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same language

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**Theorem 2.** [Mayr–ASz] *If $\mathcal{V}$ has a cube term, then* $\mathrm{SMP}(\mathcal{K}) \in \mathsf{P}$ *provided*
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- Therefore, P. Mayr's result (combined with (1)) yields a P-time algorithm.