From congruence lattices to circles and an online game (Dedicated to Professor Branimir Šešelja)

Gábor Czédli (University of Szeged)

Lecture at AAA94+NSAC 2017, Novi Sad, June 15-18

These slides: `http://www.math.u-szeged.hu/~czedli/`

June 14, 2017

How did I start to deal with Con(A) (for a single A)? or: What did I learn in Novi Sad ten years ago?

Novi Sad, 2007, invited by Branimir Šešelja and Andreja Tepavčević

Definition (Sešelja and Vojvodić, 1988)

Weak congruence: a "congruence without reflexivity"; i.e, a compatible, symmetric, transitive relation on $A = \langle A, F\rangle$. In other words: a congruence of a subalgebra or $\emptyset$. The lattice they form: $\mathrm{Cw}(A) = \langle \mathrm{Cw}(A); \subseteq\rangle$.

Šešelja and Tepavčević: Weak congruences in universal algebra, Institute of Mathematics, Novi Sad, 2001. 150 pp.

Remark

For a group G, $\mathrm{Cw}(G) \cong \mathcal{N}\mathrm{w}(G) := \{\langle X, S\rangle : X \lhd S \leq G\}$

How did I start to deal with Con(A) (for a single A)? or: What did I learn in Novi Sad ten years ago?

Novi Sad, 2007, invited by Branimir Šešelja and Andreja Tepavčević

Definition (Sešelja and Vojvodić, 1988)

Weak congruence: a "congruence without reflexivity"; i.e, a compatible, symmetric, transitive relation on $A = \langle A, F\rangle$. In other words: a congruence of a subalgebra or $\emptyset$. The lattice they form: $\mathrm{Cw}(A) = \langle \mathrm{Cw}(A); \subseteq\rangle$.

Šešelja and Tepavčević: Weak congruences in universal algebra, Institute of Mathematics, Novi Sad, 2001. 150 pp.

Remark

For a group G, $\mathrm{Cw}(G) \cong \mathcal{N}\mathrm{w}(G) := \{\langle X, S\rangle : X \triangleleft S \leq G\}$

How did I start to deal with Con(A) (for a single A)? or: What did I learn in Novi Sad ten years ago?

Novi Sad, 2007, invited by Branimir Šešelja and Andreja Tepavčević

Definition (Sešelja and Vojvodić, 1988)

Weak congruence: a “congruence without reflexivity”; i.e, a compatible, symmetric, transitive relation on $A = \langle A, F\rangle$. In other words: a congruence of a subalgebra or $\emptyset$. The lattice they form: $\mathrm{Cw}(A) = \langle \mathrm{Cw}(A); \subseteq\rangle$.

Šešelja and Tepavčević: Weak congruences in universal algebra, Institute of Mathematics, Novi Sad, 2001. 150 pp.

Remark

For a group G, $\mathrm{Cw}(G) \cong \mathcal{N}\mathrm{w}(G) := \{\langle X, S\rangle : X \lhd S \leq G\}$

How did I start to deal with Con(A) (for a single A)? or: What did I learn in Novi Sad ten years ago? 0'/50

Novi Sad, 2007, invited by Branimir Šešelja and Andreja Tepavčević

Definition (Sešelja and Vojvodić, 1988)

Weak congruence: a "congruence without reflexivity"; i.e, a compatible, symmetric, transitive relation on $A = \langle A, F \rangle$. In other words: a congruence of a subalgebra or $\emptyset$. The lattice they form: $\mathrm{Cw}(A) = \langle \mathrm{Cw}(A); \subseteq \rangle$.

Šešelja and Tepavčević: Weak congruences in universal algebra, Institute of Mathematics, Novi Sad, 2001. 150 pp.

Remark

For a group G, $\mathrm{Cw}(G) \cong \mathcal{N}\mathrm{w}(G) := \{\langle X, S \rangle : X \lhd S \leq G\}$

Characterizing Dedekind groups by Cw(G)

- **Dedekind** group $\overset{\text{def}}{\iff}$ $\forall$ subgroups are normal. (Hamiltonian group $\overset{\text{def}}{\iff}$ Dedekind & non-abelian.)
- If $\forall x, y \in$ a lattice, $d \vee x = d \vee y \Rightarrow d \vee x = d \vee (x \wedge y)$, then d is a join-semidistributive element.
- $\Delta := \{\langle x, x\rangle : x \in A\} \in \mathrm{Cw}(A)$, $\Delta := \langle\{1\}, G\rangle \in \mathcal{N}\mathrm{w}(G)$.
- Czédli, Šešelja, Tepavčević: On the semidistributivity of elements in weak congruence lattices of algebras and groups, Algebra Universalis 58 (2008), 349–355 ; it includes:

Theorem (Czédli, Šešelja, Tepavčević; AU2008)

For every finite group G, the following are equivalent

1. *G is a Dedekind group;*
2. *Δ is a join-semidistributive element in $\mathrm{Cw}(G)$;*
3. *Δ is a join-semidistributive element in $\mathcal{N}\mathrm{w}(G)$.*

Characterizing Dedekind groups by $\mathrm{Cw}(G)$

- **Dedekind** group $\overset{\text{def}}{\iff}$ $\forall$ subgroups are normal. (Hamiltonian group $\overset{\text{def}}{\iff}$ Dedekind & non-abelian.)
- If $\forall x, y \in$ a lattice, $d \vee x = d \vee y \Rightarrow d \vee x = d \vee (x \wedge y)$, then d is a **join-semidistributive element**.
- $\Delta := \{\langle x, x\rangle : x \in A\} \in \mathrm{Cw}(A)$, $\quad \Delta := \langle\{1\}, G\rangle \in \mathcal{N}\mathrm{w}(G)$.
- Czédli, Šešelja, Tepavčević: On the semidistributivity of elements in weak congruence lattices of algebras and groups, Algebra Universalis 58 (2008), 349–355 ; it includes:

Theorem (Czédli, Šešelja, Tepavčević; AU2008)

For every finite group G, the following are equivalent

1. *G is a Dedekind group;*
2. *Δ is a join-semidistributive element in $\mathrm{Cw}(G)$;*
3. *Δ is a join-semidistributive element in $\mathcal{N}\mathrm{w}(G)$.*

Characterizing Dedekind groups by Cw(G)

- **Dedekind** group $\overset{\text{def}}{\iff}$ $\forall$ subgroups are normal. (Hamiltonian group $\overset{\text{def}}{\iff}$ Dedekind & non-abelian.)
- If $\forall x, y \in$ a lattice, $d \vee x = d \vee y \Rightarrow d \vee x = d \vee (x \wedge y)$, then d is a **join-semidistributive element**.
- $\Delta := \{\langle x, x\rangle : x \in A\} \in \mathrm{Cw}(A)$, $\quad \Delta := \langle\{1\}, G\rangle \in \mathcal{N}\mathrm{w}(G)$.
- Czédli, Šešelja, Tepavčević: On the semidistributivity of elements in weak congruence lattices of algebras and groups, Algebra Universalis 58 (2008), 349–355 ; it includes:

Theorem (Czédli, Šešelja, Tepavčević; AU2008)

For every finite group G, the following are equivalent

1. *G is a Dedekind group;*
2. *Δ is a join-semidistributive element in $\mathrm{Cw}(G)$;*
3. *Δ is a join-semidistributive element in $\mathcal{N}\mathrm{w}(G)$.*

$D \cong \mathrm{Con}(L)$ and find “nice” L !

Theorem (Funayama & Nakayama 1942, Dilworth, Wehrung 2007)

For every lattice L, $\mathrm{Con}(L)$ is distributive. Conversely:

- *if D is a finite distributive lattice, then $\exists L$ s.t. $D \cong \mathrm{Con}(L)$.*
- *F. Wehrung (2007): in general (without finiteness) the converse fails!*

Find **nice** L! *G. Grätzer: The congruences of a finite lattice. A "proof-by-picture” approach. 2nd ed., 2016.* E.g.:

Definition (G. Grätzer and E. Knapp 2009)

A lattice L is **rectangular** if the following conditions hold:

- L is planar (and so finite by definition) and $|L| > 1$,
- L is semimodular, i.e, $\forall x, y, z$, if $x \preceq y$ then $x \vee z \preceq y \vee z$,
- $\exists u, v$ doubly irreducible s.t. $u \vee v = 1$ and $u \wedge v = 0$.

$D \cong \mathrm{Con}(L)$ and find “nice” L !

Theorem (Funayama & Nakayama 1942, Dilworth, Wehrung 2007)

For every lattice L, $\mathrm{Con}(L)$ *is distributive. Conversely:*

- *if D is a finite distributive lattice, then $\exists L$ s.t. $D \cong \mathrm{Con}(L)$.*
- *F. Wehrung (2007): in general (without finiteness) the converse fails!*

Find **nice** L! *G. Grätzer: The congruences of a finite lattice. A "proof-by-picture” approach. 2nd ed., 2016.* E.g.:

Definition (G. Grätzer and E. Knapp 2009)

A lattice L is **rectangular** if the following conditions hold:

- L is planar (and so finite by definition) and $|L| > 1$,
- L is semimodular, i.e, $\forall x, y, z$, if $x \preceq y$ then $x \vee z \preceq y \vee z$,
- $\exists u, v$ doubly irreducible s.t. $u \vee v = 1$ and $u \wedge v = 0$.

$D \cong \text{Con}(L)$ and find "nice" L !

Theorem (Funayama & Nakayama 1942, Dilworth, Wehrung 2007)

For every lattice L, $\text{Con}(L)$ is distributive. Conversely:

- *if D is a finite distributive lattice, then $\exists L$ s.t. $D \cong \text{Con}(L)$.*
- *F. Wehrung (2007): in general (without finiteness) the converse fails!*

Find **nice** L! *G. Grätzer: The congruences of a finite lattice. A "proof-by-picture" approach. 2nd ed., 2016.* E.g.:

Definition (G. Grätzer and E. Knapp 2009)

A lattice L is **rectangular** if the following conditions hold:

- L is planar (and so finite by definition) and $|L| > 1$,
- L is semimodular, i.e, $\forall x, y, z$, if $x \preceq y$ then $x \vee z \preceq y \vee z$,
- $\exists u, v$ doubly irreducible s.t. $u \vee v = 1$ and $u \wedge v = 0$.

Nice L's are found. How nice?

Theorem (G. Grätzer and E. Knapp 2009)

*For every finite distributive D with $|D| > 1$, $\exists L$ **rectangular** s.t. $D \cong \mathrm{Con}(L)$.*

This is not a mathematical theorem!

Fur this purpose, there are no nicer lattices than rectangular.

Sketch of "proof".

- L modular $\Rightarrow$ $\mathrm{Con}(L)$ is boolean.
- L is nonplanar $\Rightarrow$ diagrams are difficult to read.
- L is only planar and semimodular (G. Grätzer, H. Lakser and E. T. Schmidt, 1998): ragged diagram.

☐

Nice L's are found. How nice?

Theorem (G. Grätzer and E. Knapp 2009)

For every finite distributive D with $|D| > 1$, $\exists L$ ***rectangular*** *s.t. $D \cong \mathrm{Con}(L)$.*

This is not a mathematical theorem!

Fur this purpose, there are no nicer lattices than rectangular.

Sketch of "proof".

- L modular $\Rightarrow$ $\mathrm{Con}(L)$ is boolean.
- L is nonplanar $\Rightarrow$ diagrams are difficult to read.
- L is only planar and semimodular (G. Grätzer, H. Lakser and E. T. Schmidt, 1998): ragged diagram.

☐

Nice L's are found. How nice?

Theorem (G. Grätzer and E. Knapp 2009)

For every finite distributive D with $|D| > 1$, $\exists L$ ***rectangular*** *s.t. $D \cong \mathrm{Con}(L)$.*

This is not a mathematical theorem!

Fur this purpose, there are no nicer lattices than rectangular.

Sketch of "proof".

- L modular $\Rightarrow$ $\mathrm{Con}(L)$ is boolean.
- L is nonplanar $\Rightarrow$ diagrams are difficult to read.
- L is only planar and semimodular (G. Grätzer, H. Lakser and E. T. Schmidt, 1998): ragged diagram.

☐

The structure of rectangular (and related) lattices

Definition (G. Grätzer and E. Knapp 2007)

A planar semimodular lattice L is **slim** if it contains no M_3 $\iff$ it contains no cover-preserving M_3.

Grid: direct product of two finite chains (with the usual diagram).

Before formulating a structure theorem, we need to define three constructing steps.

The structure of rectangular (and related) lattices

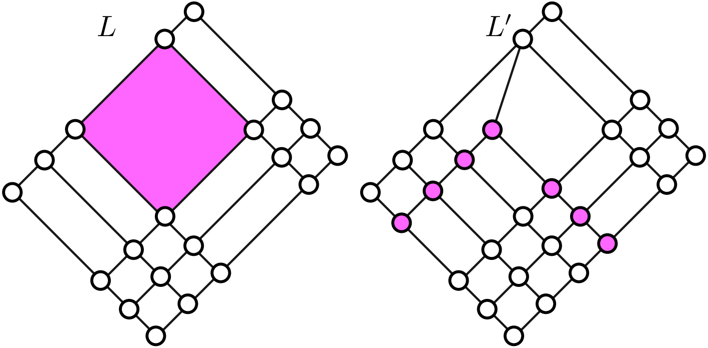
Definition (G. Grätzer and E. Knapp 2007)

A planar semimodular lattice L is **slim** if it contains no M_3 $\iff$ it contains no cover-preserving M_3.

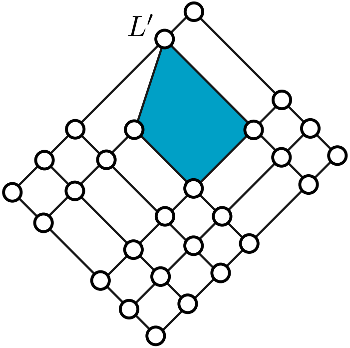
Grid: direct product of two finite chains (with the usual diagram).

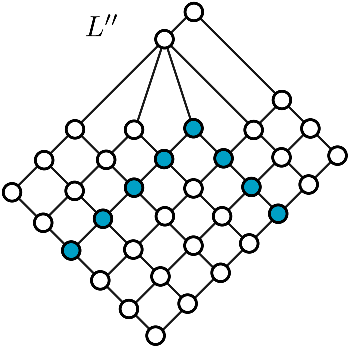
Before formulating a structure theorem, we need to define three constructing steps.

Inserting a fork

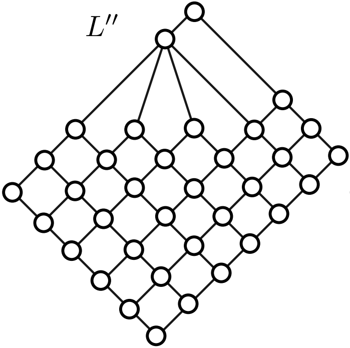


Inserting another fork

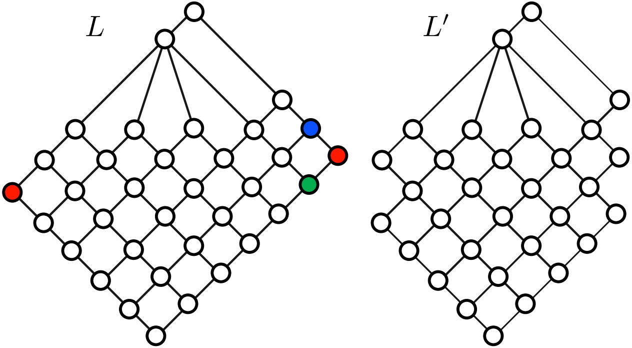




Inserting eyes (to turn *slim* to *non-slim*)



Deleting corners (1st red, 2nd green, 3rd blue)



Structure theorems (for rectangular and related)

Theorem (Czédli and E.T. Schmidt, 2012)

A lattice L is rectangular iff it can be obtained from a grid by

1. *inserting forks (finitely many, one by one) first, and*
2. *inserting eyes (finitely many, one by one)*

Note: (1) and (2) do not commute (it would not make sense) !

Theorem (Czédli and E.T. Schmidt, 2012)

Planar and semimodular $\iff$ *obtained from a grid by*

1. *inserting forks (finitely many, one by one),*
2. *deleting corners (finitely many, one by one), and*
3. *inserting eyes (finitely many, one by one)*

in this order.

Note: we cannot change the order of (1), (2), and (3) in general.

Structure theorems (for rectangular and related)

Theorem (Czédli and E.T. Schmidt, 2012)

A lattice L is rectangular iff it can be obtained from a grid by

1. *inserting forks (finitely many, one by one) first, and*
2. *inserting eyes (finitely many, one by one)*

Note: (1) and (2) do not commute (it would not make sense) !

Theorem (Czédli and E.T. Schmidt, 2012)

Planar and semimodular $\iff$ *obtained from a grid by*

1. *inserting forks (finitely many, one by one),*
2. *deleting corners (finitely many, one by one), and*
3. *inserting eyes (finitely many, one by one)*

in this order.

Note: we cannot change the order of (1), (2), and (3) in general.

The 2012 Thm. for two rectangular lattices

Grätzer–Knapp 2009: $(\forall D \text{ with } |D| > 1)\ (\exists K \text{ rect})\ (D \cong \mathrm{Con}(K))$.

Theorem (Czédli, AU Algebra **67**, 313–345, 2012)

$\forall$ BLACK part, $\exists$ a RED part s.t. the diagram commutes:

$$\begin{tikzcd}
D \arrow[rr, "\varphi", "\{0,1\}\text{-preserving lattice homomorphism}"'] \arrow[d, dotted, "\cong"'] & & \mathrm{Con}(L) \\
\mathrm{Con}(K) \arrow[urr, dotted, bend right, "\text{restriction}"] & &
\end{tikzcd}$$

D finite distr.

L is rectangular

K is rectangular and L is a rectangular filter of K

The 2012 Thm. implies the 2009 Thm. as follows.

$|\mathrm{Con}(M_3)| = 2$. The Prime Ideal Theorem yields a $\varphi\colon D \to \mathrm{Con}(M_3)$. M_3 is rectangular. The 2012 Thm. gives a K. Thus, the 2009 Thm. follows. ☐

The 2012 Thm. for two rectangular lattices

Grätzer–Knapp 2009: $(\forall D \text{ with } |D| > 1)\ (\exists K \text{ rect})\ (D \cong \mathrm{Con}(K))$.

Theorem (Czédli, AU Algebra **67**, 313–345, 2012)

∀ BLACK part, ∃ a RED part s.t. the diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D \arrow[r, "\varphi"] \arrow[d, dashed, "\cong"'] & \mathrm{Con}(L) \\
\mathrm{Con}(K) \arrow[ur, dashed, bend right, "\text{restriction}"] &
\end{tikzcd}$$

finite distr.

$\{0,1\}$-preserving lattice homomorphism

L is rectangular

K is rectangular and L is a rectangular filter of K

The 2012 Thm. implies the 2009 Thm. as follows.

$|\mathrm{Con}(M_3)| = 2$. The Prime Ideal Theorem yields a $\varphi\colon D \to \mathrm{Con}(M_3)$. M_3 is rectangular. The 2012 Thm. gives a K. Thus, the 2009 Thm. follows. ☐

The 2012 Thm. for two rectangular lattices

Grätzer–Knapp 2009: $(\forall D \text{ with } |D| > 1)\ (\exists K \text{ rect})\ (D \cong \mathrm{Con}(K))$.

Theorem (Czédli, AU Algebra **67**, 313–345, 2012)

$\forall$ BLACK part, $\exists$ a RED part s.t. the diagram commutes:

$$\begin{tikzcd}
D \arrow[rr, "\varphi"] \arrow[d, dotted, "\cong"'] & & \mathrm{Con}(L) \\
\mathrm{Con}(K) \arrow[urr, dotted, bend right, "\text{restriction}"] & &
\end{tikzcd}$$

D finite distr.

$\{0,1\}$-preserving lattice homomorphism

L is rectangular

K is rectangular and L is a rectangular filter of K

The 2012 Thm. implies the 2009 Thm. as follows.

$|\mathrm{Con}(M_3)| = 2$. The Prime Ideal Theorem yields a $\varphi\colon D \to \mathrm{Con}(M_3)$. M_3 is rectangular. The 2012 Thm. gives a K. Thus, the 2009 Thm. follows. ☐

Rectangular representation of $\{0,1\}$-homomorphisms 20'/30

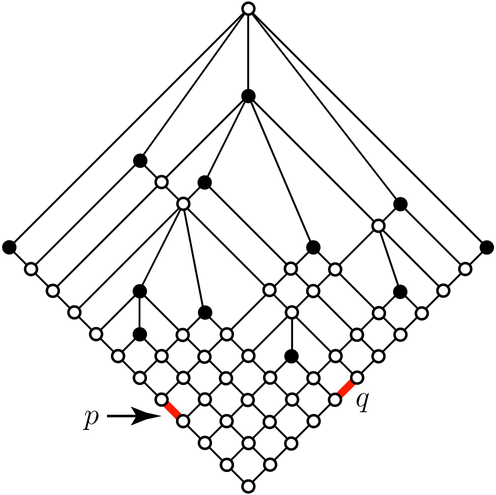
Corollary (Czédli, AU Algebra **67**, 313–345, 2012)

Whatever the BLACK part is, there exists a RED part:

$$\begin{tikzcd}
\begin{matrix} D \\ \text{finite distr.} \end{matrix} \arrow[rr, "\varphi", "\begin{matrix}\{0,1\}\text{-preserving} \\ \text{lattice homomorphism}\end{matrix}"'] \arrow[d, dotted, "\cong"'] & & \begin{matrix} E \\ \text{finite distr.} \end{matrix} \\
\mathrm{Con}(K) \arrow[rr, dotted, "\text{restriction}"] & & \mathrm{Con}(L) \arrow[u, dotted, "\cong"']
\end{tikzcd}$$

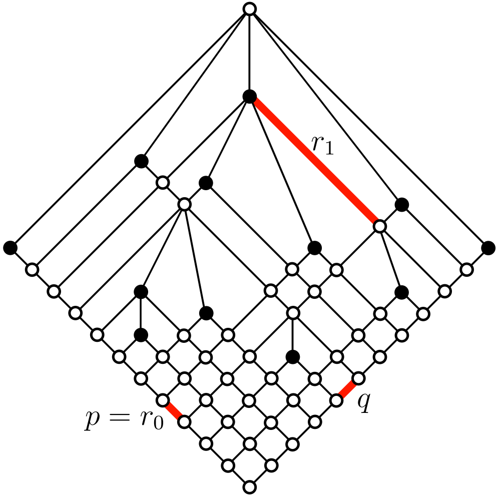
K is rectangular and L is a rectangular filter of K *such that the diagram commutes.*

Toolkit $\ni$ Grätzer's Swing Lemma

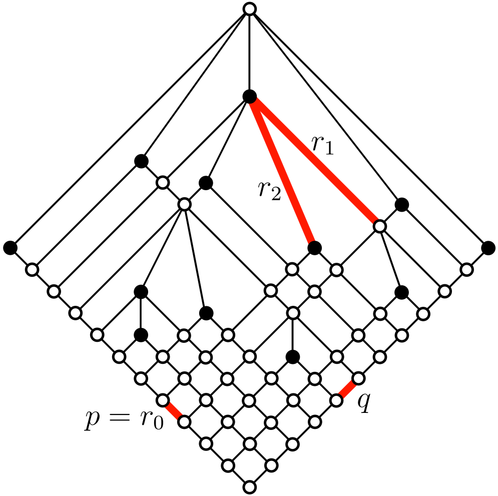


Swing Lemma (Grätzer, 2015): Let L be slim semimodular.
con(p) $\geq$ con(q) iff $\exists$ a sequence of edges as shown in the figures.

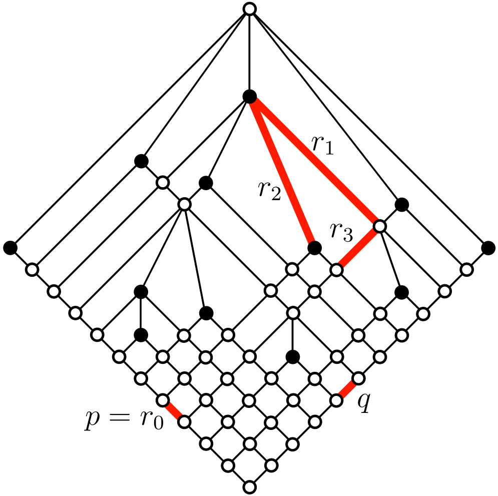
Jump to an opposite edge (7 times)



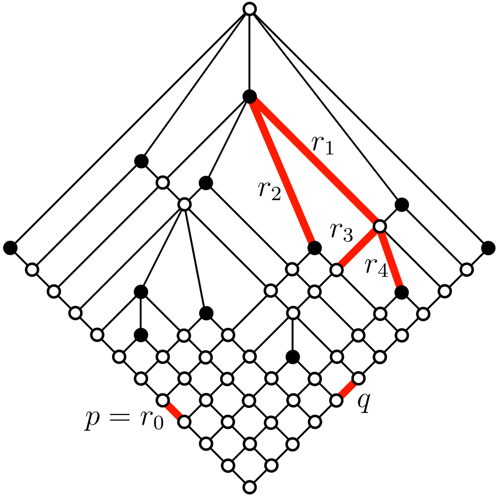
Swing (to a neighboring edge hanging inside)



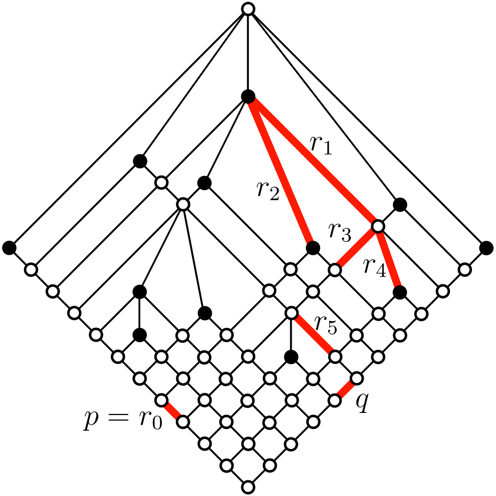
Jump to an opposite edge (once)



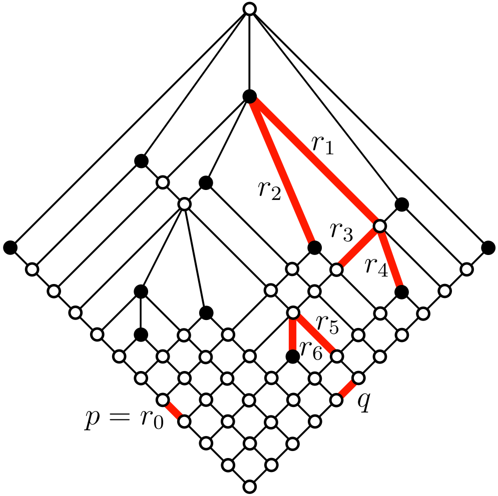
Swing



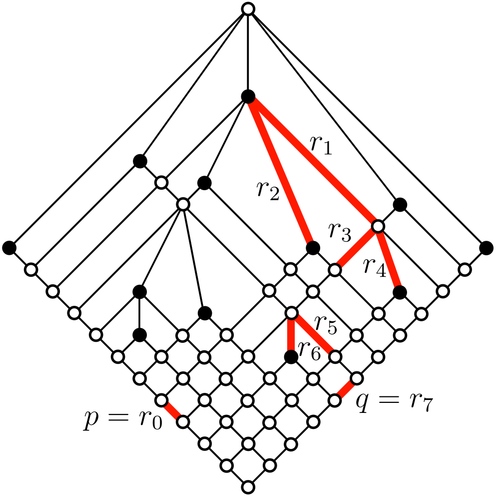
Jump to an opposite edge (3 times)



Swing



Jump to an opposite edge (2 times); arrival



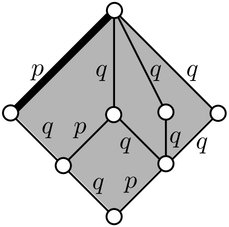
First excursion outside Lattice Theory and even outside Mathematics

Swing Lattice Game (Czédli and Géza Makay, 2016)

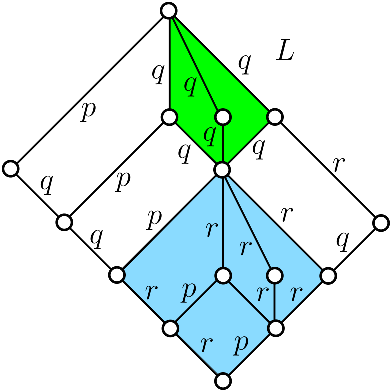
See http:// www.math.u-szeged.hu/~makay/swinglattice/

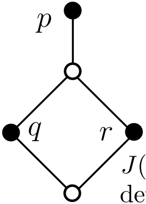
or my web-site; instructions are given there. In addition of jumping and swinging, tilting (the dual of swinging) is also allowed. L is almost slim and semimodular; only one eye is added and its position is controlled by the player.

The main idea of Grätzer and Knapp's proof



$S_8(p \geq q)$

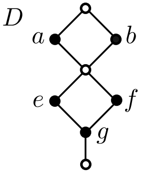


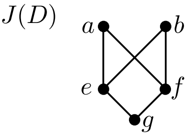


$J(D)$, which determines D

Representing D with a rectangular L such that $D \cong \mathrm{Con}(L)$.

Can we do it without eyes, i.e., without M_3's?





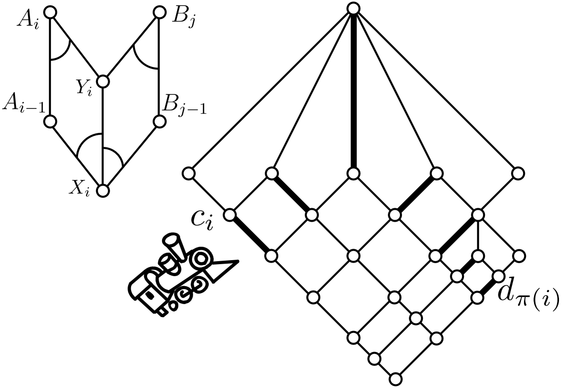
Theorem (Czédli, AU 72 225–230, 2014)

*This D **cannot** be represented as* Con(L) *with a slim planar semimodular L.*

That is, eyes are really needed in general.

2nd excursion outside Lat.Th.: Jordan–Hölder **1889**

∃ down-and-up matching: Grätzer–Nation, AU **64** 309-311, 2011.

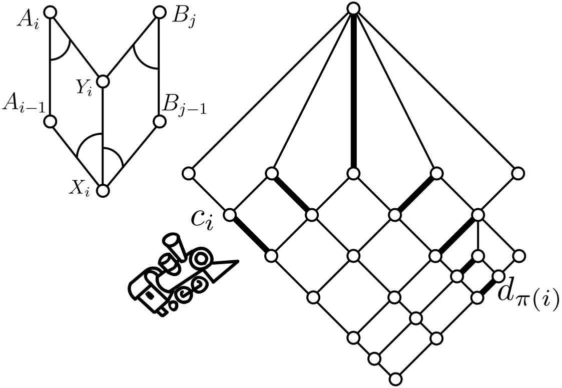


Theorem (Czédli and Schmidt, AU 66, 69-79 (2011))

Between any two composition series of a finite group, there is exactly one down-and-up matching.

2nd excursion outside Lat.Th.: Jordan–Hölder **1889** 31'/19

∃ down-and-up matching: Grätzer–Nation, AU **64** 309-311, 2011.



Theorem (Czédli and Schmidt, AU **66**, 69-79 (2011))

Between any two composition series of a finite group, there is exactly one down-and-up matching.

Enumerating results

Remember: **slim rectangular** = rectangular without M_3 = what we can obtain from a grid by inserting forks.

Theorem (Czédli, Dékány, Gyenizse, and Kulin, AU 75 33-50, 2016)

The number of slim rectangular lattices of length n is asymptotically $(n-2)! \cdot e^2/2$, *where* $e = \lim((1+1/n)^n) \approx 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,35\ldots$

Similar results for "slim semimodular" instead of "slim rectangular":

- Czédli, Ozsvárt, Udvari (Discr. Math. **312** 3523–3536, 2012),
- Czédli, Dékány, Ozsvárt, Szakács, Udvari (Math. Slovaca **66** 2016, 5–18),
- Czédli (Order **33** 2016, 231–237),

without e. (Yet another aspect of rectangular ⇒ nice.)

Enumerating results

Remember: **slim rectangular** = rectangular without M_3 = what we can obtain from a grid by inserting forks.

Theorem (Czédli, Dékány, Gyenizse, and Kulin, AU 75 33-50, 2016)

The number of slim rectangular lattices of length n is asymptotically $(n-2)! \cdot e^2/2$, *where*
$e = \lim((1+1/n)^n) \approx 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,35\ldots$

Similar results for "slim semimodular" instead of "slim rectangular":

- Czédli, Ozsvárt, Udvari (Discr. Math. **312** 3523–3536, 2012),
- Czédli, Dékány, Ozsvárt, Szakács, Udvari (Math. Slovaca **66** 2016, 5–18),
- Czédli (Order **33** 2016, 231–237),

without e. (Yet another aspect of rectangular ⇒ nice.)

A link to convex geometries

Definition (Edelman 1980; Adaricheva, Gorbunov, Tumanov 2003)

A pair $\langle E, \Phi\rangle$ is a *convex geometry*, also called *anti-exchange system*, if $\Phi\colon P(E) \to P(E)$ is a closure operator, $\Phi(\emptyset) = \emptyset$, and

- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, and we have $\Phi(A \cup \{x\}) = \Phi(A \cup \{y\})$, then $x = y$. (Anti-exchange property; x cannot be exchanged to y.)

Theorem (Czédli, Discrete Mathematics 330, 61-75, 2014)

Each slim semimodular $L \cong_{\text{dual}} \text{Sub}($*a conv. geom. of circles*$)$.

Are circles sufficient to represent all convex geometries in this way?

Thm (Adaricheva–Bolat 2016, https://arxiv.org/abs/1609.00092)

No, because circles satisfy property (3) of the next slide!

A link to convex geometries

Definition (Edelman 1980; Adaricheva, Gorbunov, Tumanov 2003)

A pair $\langle E, \Phi\rangle$ is a *convex geometry*, also called *anti-exchange system*, if $\Phi\colon P(E) \to P(E)$ is a closure operator, $\Phi(\emptyset) = \emptyset$, and

- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, and we have $\Phi(A \cup \{x\}) = \Phi(A \cup \{y\})$, then $x = y$. (Anti-exchange property; x cannot be exchanged to y.)

Theorem (Czédli, Discrete Mathematics 330, 61-75, 2014)

Each slim semimodular $L \cong_{\text{dual}} \operatorname{Sub}($*a conv. geom. of circles*$)$.

Are circles sufficient to represent all convex geometries in this way?

Thm (Adaricheva–Bolat 2016, https://arxiv.org/abs/1609.00092)

No, because circles satisfy property (3) of the next slide!

A link to convex geometries

Definition (Edelman 1980; Adaricheva, Gorbunov, Tumanov 2003)

A pair $\langle E, \Phi\rangle$ is a *convex geometry*, also called *anti-exchange system*, if $\Phi\colon P(E) \to P(E)$ is a closure operator, $\Phi(\emptyset) = \emptyset$, and

- If $\Phi(A) = A \in P(E)$, $x, y \in E$, $x \notin A$, $y \notin A$, and we have $\Phi(A \cup \{x\}) = \Phi(A \cup \{y\})$, then $x = y$. (Anti-exchange property; x cannot be exchanged to y.)

Theorem (Czédli, Discrete Mathematics 330, 61-75, 2014)

Each slim semimodular $L \cong_{\text{dual}} \mathrm{Sub}($*a conv. geom. of circles*$)$.

Are circles sufficient to represent all convex geometries in this way?

Thm (Adaricheva–Bolat 2016, https://arxiv.org/abs/1609.00092)

No, because circles satisfy property (3) of the next slide!

A characterization of circles

Theorem (K. Adaricheva and M. Bolat, 2016)

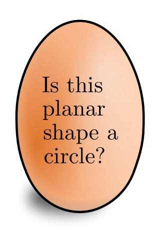
Circles in the plane satisfy the convex combinatorial property described below in (3) (even if A_0, A_1, A_2 are circles.)

Theorem (Czédli 2016, `http://arxiv.org/abs/1611.09331`)

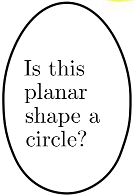
If K_0 is a compact convex subset of the plane $\mathbb{R}^2$, then tfae:.

1. *K_0 is a disk.*
2. *For every $K_1 \subseteq \mathbb{R}^2$ and for arbitrary points $A_0, A_1, A_2 \in \mathbb{R}^2$, if K_1 is* ***isometric*** *to K_0 and both K_0 and K_1 are subsets of the triangle $\mathrm{Conv}(\{A_0, A_1, A_2\})$, then there exist a $j \in \{0, 1, 2\}$ and an $i \in \{0, 1\}$ such that K_{1-i} is a subset of $\mathrm{Conv}\big(K_i \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big)$.*
3. *The same as the second condition but "isometric" is replaced by "similar". (This is the* ***Adaricheva–Bolat property****.)*

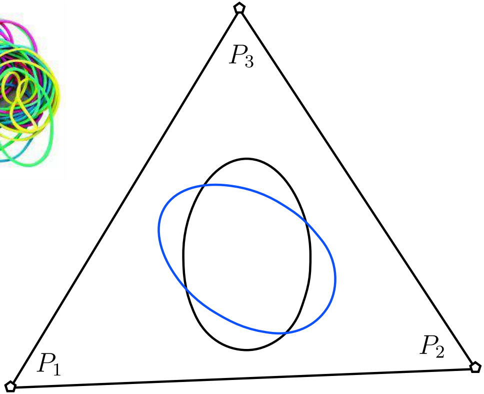
3rd excursion (guided tour for non-mathematicians)



Is this planar shape a circle?

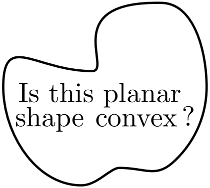


Is this planar shape a circle?

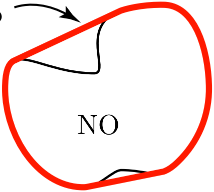


Guided tour for non-mathematicians/2

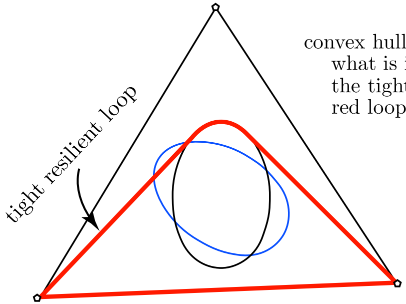
convex hull = what is inside
the tight resilient red loop



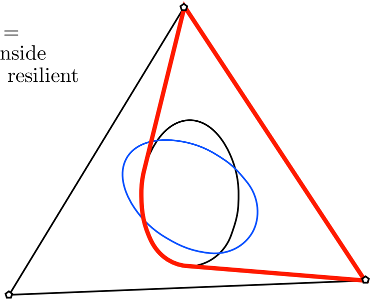
Is this planar
shape convex ?



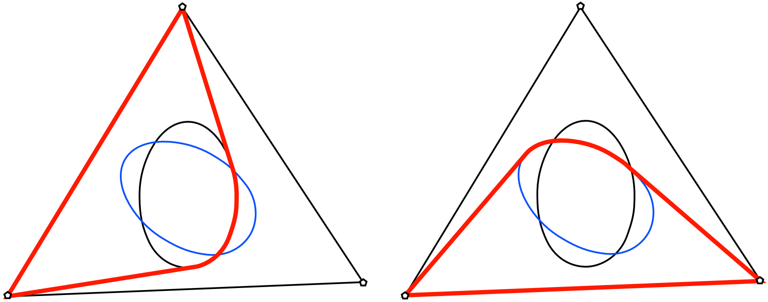
Guided tour for non-mathematicians/3



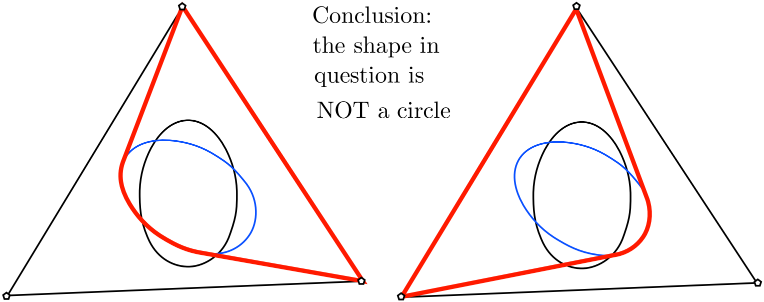
convex hull =
what is inside
the tight resilient
red loop



Guided tour for non-mathematicians/4



Guided tour for non-mathematicians/5



Conclusion:
the shape in
question is
NOT a circle

Now, something deeper: Comp(L) . . .

An algebraic lattice is *strongly atomic* if each of its non-singleton intervals contains an atom (with respect to the interval).

Definition (A. P. Huhn's *n*-distributivity)

$$x \wedge \bigvee_{0 \le i \le n} y_i = \bigvee_{0 \le j \le n} \Bigl(x \wedge \bigvee_{0 \le i \le n,\ i \ne j} y_i \Bigr)$$

(Note: distributivity = 1-distributivity.)

Theorem (R. Freese, G. Grätzer, E.T. Schmidt 1991; Grätzer and E.T. Schmidt 2001)

Every complete lattice A is isomorphic to Comp(K) *for a suitable strongly atomic, 3-distributive, complete modular lattice K (1991). The same holds with "2-distributive" (2001).*

3 ↘ 2: ten years. 2 ↘ 1?: we know that this is impossible

Now, something deeper: Comp(L) . . .

An algebraic lattice is *strongly atomic* if each of its non-singleton intervals contains an atom (with respect to the interval).

Definition (A. P. Huhn's *n*-distributivity)

$$x \wedge \bigvee_{0\leq i\leq n} y_i = \bigvee_{0\leq j\leq n}\Big(x \wedge \bigvee_{0\leq i\leq n,\ i\neq j} y_i\Big)$$

(Note: distributivity = 1-distributivity.)

Theorem (R. Freese, G. Grätzer, E.T. Schmidt 1991; Grätzer and E.T. Schmidt 2001)

Every complete lattice A is isomorphic to Comp(K) *for a suitable strongly atomic, 3-distributive, complete modular lattice K (1991). The same holds with "2-distributive" (2001).*

$3 \searrow 2$: ten years. $2 \searrow 1$?: we know that this is impossible

Main Theorem

Main Theorem (1st part) (Czédli, AU, to appear)

∀ BLACK, ∃ RED such that the following diagram commutes:

$$\begin{tikzcd}
\underset{\text{complete lattice}}{A} \arrow[r, "f"] \arrow[r, swap, "\substack{(\bigvee, 0)\text{-preserving} \\ \text{and } 0\text{-separating}}"] \arrow[d, "\cong"'] & \underset{\text{complete lattice}}{A'} \\
\mathrm{Com}(K) \arrow[r, "\text{extension}"] & \mathrm{Com}(K') \arrow[u, "\cong"']
\end{tikzcd}$$

K and K' are complete, modular, strongly atomic, and 2-distributive, and K is a principal filter of K'

Of course, we could say "ideal" instead of "filter"

Main Theorem

Main Theorem (2nd part) (Czédli, AU, to appear)

∀ BLACK, ∃ RED such that the following diagram commutes:

$$\begin{tikzcd}
\underset{\text{complete lattice}}{A} \arrow[r, "f", "{(\bigwedge, 0, 1)\text{-preserving}}"'] \arrow[d, "\cong"'] & \underset{\text{complete lattice}}{A'} \\
\mathrm{Com}(K) \arrow[r, "\text{restriction}"] & \mathrm{Com}(K') \arrow[u, "\cong"']
\end{tikzcd}$$

K and K' are complete, modular, strongly atomic, and 2-distributive, and K' is a sublattice of K

Main Theorem

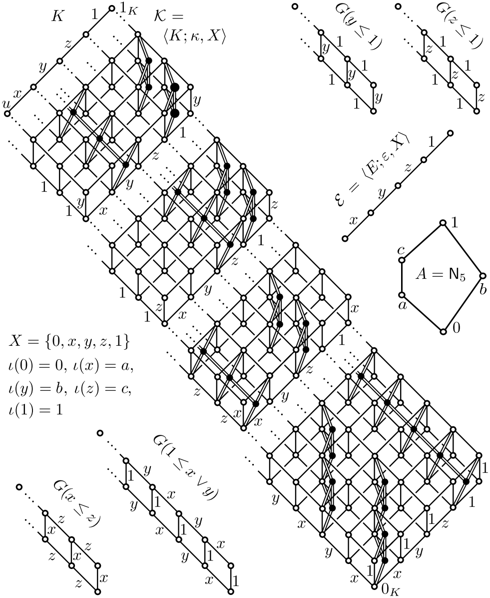
Main Theorem (3rd part) (Czédli, AU, to appear)

∀ BLACK, ∃ RED such that the following diagram commutes:

$$\begin{tikzcd}
\underset{\text{complete lattice}}{A} \arrow[r, "f", "(\bigvee, 0)\text{-preserving}"'] \arrow[d, "\cong"'] & \underset{\text{complete lattice}}{A'} \\
\mathrm{Com}(K) \arrow[r, "\text{the composite of } \varphi \text{ and generation}"] & \mathrm{Com}(K') \arrow[u, "\cong"']
\end{tikzcd}$$

K and K' are complete, modular, strongly atomic, and 2-distributive, and $\varphi : K \to K'$ is a complete-lattice homomorphism

Construction for $A := N_5$ (only for a single lattice)



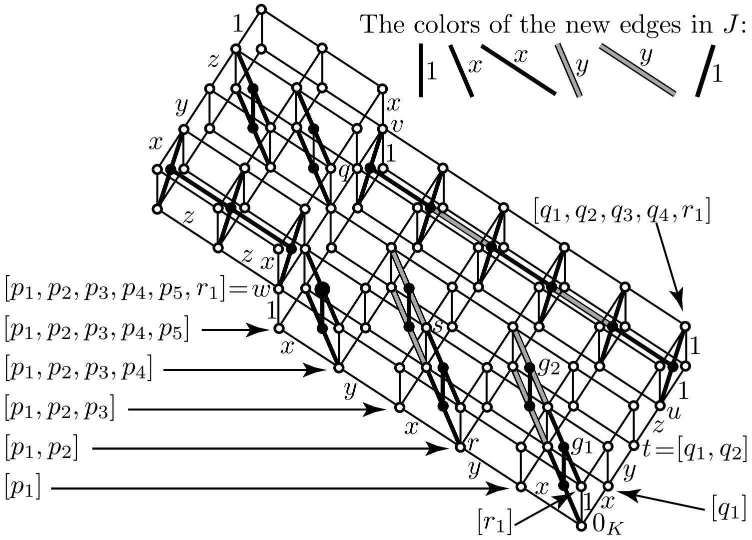
$X = \{0, x, y, z, 1\}$

$\iota(0) = 0,\ \iota(x) = a,$

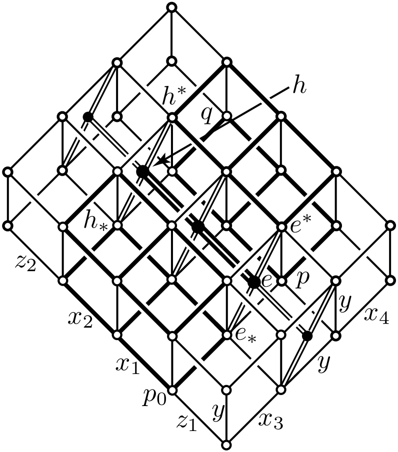
$\iota(y) = b,\ \iota(z) = c,$

$\iota(1) = 1$

Why does K have the required properties?



How do congruences look like? (illustration)



The poset of principal lattice congruences

Theorem (Grätzer, 2013)

For every poset Q with 0 and 1, there is a lattice L such that $\mathrm{Princ}(L)$*, the poset of principal congruences of L with respect to $\subseteq$, is isomorphic to Q.*

The $1 \notin Q$ case is much harder and it is still unsolved. The problem is solved only up to $\aleph_0$. Namely:

Theorem (Czédli, AU 75 (2016), 351–380)

Let Q be a poset with $|Q| \le \aleph_0$. Then Q is representable as $\mathrm{Princ}(L)$ *for some lattice L iff $0 \in Q$ and Q is directed.*

The poset of principal lattice congruences

Theorem (Grätzer, 2013)

For every poset Q with 0 and 1, there is a lattice L such that $\mathrm{Princ}(L)$, the poset of principal congruences of L with respect to $\subseteq$, is isomorphic to Q.

The $1 \notin Q$ case is much harder and it is still unsolved. The problem is solved only up to $\aleph_0$. Namely:

Theorem (Czédli, AU 75 (2016), 351–380)

Let Q be a poset with $|Q| \leq \aleph_0$. Then Q is representable as $\mathrm{Princ}(L)$ for some lattice L iff $0 \in Q$ and Q is directed.

The poset of principal lattice congruences

Theorem (Grätzer, 2013)

For every poset Q with 0 and 1, there is a lattice L such that $\mathrm{Princ}(L)$, *the poset of principal congruences of L with respect to $\subseteq$, is isomorphic to Q.*

The $1 \notin Q$ case is much harder and it is still unsolved. The problem is solved only up to $\aleph_0$. Namely:

Theorem (Czédli, AU 75 (2016), 351–380)

Let Q be a poset with $|Q| \leq \aleph_0$. Then Q is representable as $\mathrm{Princ}(L)$ *for some lattice L iff $0 \in Q$ and Q is directed.*

A by-product of studying Princ(L)

Every map, say, $f\colon A \to B$, can be decomposed as a composite of a surjective map and an injective one. Namely, $f = f_2 \circ f_1$, where $f_1\colon A \to A/\mathrm{Ker}(f_1)$ is surjective, and f_2 is injective. But can we do this **naturally** for a whole family of maps? In general, **no, we cannot**. But sometimes we can do something similar.

For categories **A** and **B**, if **A** is subcategory of **B**, then the **embedding functor** acting identically will be denoted by $\iota_{\mathbf{A},\mathbf{B}}$.

A category is **concrete** if its objects are sets, possibly with structures on them, its morphisms are maps, and the composition in the category is that of maps in the usual sense. The category of sets with all maps is denoted by **Set**.

A by-product of studying Princ(L)

Every map, say, $f\colon A \to B$, can be decomposed as a composite of a surjective map and an injective one. Namely, $f = f_2 \circ f_1$, where $f_1\colon A \to A/\mathrm{Ker}(f_1)$ is surjective, and f_2 is injective. But can we do this **naturally** for a whole family of maps? In general, **no, we cannot**. But sometimes we can do something similar.

For categories $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A}$ is subcategory of $\mathbf{B}$, then the **embedding functor** acting identically will be denoted by $\iota_{\mathbf{A},\mathbf{B}}$.

A category is **concrete** if its objects are sets, possibly with structures on them, its morphisms are maps, and the composition in the category is that of maps in the usual sense. The category of sets with all maps is denoted by **Set**.

A by-product of studying Princ(L)

Every map, say, $f\colon A \to B$, can be decomposed as a composite of a surjective map and an injective one. Namely, $f = f_2 \circ f_1$, where $f_1\colon A \to A/\mathrm{Ker}(f_1)$ is surjective, and f_2 is injective. But can we do this **naturally** for a whole family of maps? In general, **no, we cannot**. But sometimes we can do something similar.

For categories $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A}$ is subcategory of $\mathbf{B}$, then the **embedding functor** acting identically will be denoted by $\iota_{\mathbf{A},\mathbf{B}}$.

A category is **concrete** if its objects are sets, possibly with structures on them, its morphisms are maps, and the composition in the category is that of maps in the usual sense. The category of sets with all maps is denoted by **Set**.

Cometic functor

Theorem (See my home-page: "Cometic functors . . . ", 2016)

For every small concrete category $\mathbf{A}$*, there exists a so-called* ***cometic functor*** $F_{\mathrm{com}}\colon \mathbf{A} \to \mathbf{Set}$ *and a natural transformation* $\pi^{\mathrm{com}}\colon F_{\mathrm{com}} \to \iota_{\mathbf{A},\mathbf{Set}}$ *such that the components of* π^{com} *are surjective maps and the* F_{com}*-image of every* ***mono****morphisms of* $\mathbf{A}$ *is an injective map.*

An illustration of the cometic functor

$$\begin{tikzcd}
X \arrow[r, "g"] & Y \\
F_{\mathrm{com}}(X) \arrow[r, "F_{\mathrm{com}}(g)"'] \arrow[d, "\boldsymbol{\pi}^{\mathrm{com}}_X"] & F_{\mathrm{com}}(Y) \arrow[d, "\boldsymbol{\pi}^{\mathrm{com}}_Y"'] \\
\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(X) \arrow[r, "\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(g)"] & \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(Y)
\end{tikzcd}$$

Here X, Y are objects and g is a morphism of $\mathbf{A}$. Their F_{com}-images are sets and a map, respectively. While there can be non-injective monomorphisms in $\mathbf{A}$, this cannot happen in $F_{\mathrm{com}}(\mathbf{A})$. Thus, $F_{\mathrm{com}}(\mathbf{A})$ is *nice* in the sense that injective ⇔ mono. Furthermore, the vertical arrows are onto maps.

An illustration of the cometic functor

$$\begin{tikzcd}
X \arrow[r, "g"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
F_{\mathrm{com}}(X) \arrow[r, "F_{\mathrm{com}}(g)"'] \arrow[d, "\boldsymbol{\pi}_X^{\mathrm{com}}"] & F_{\mathrm{com}}(Y) \arrow[d, "\boldsymbol{\pi}_Y^{\mathrm{com}}"'] \\
\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(X) \arrow[r, "\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(g)"] & \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(Y)
\end{tikzcd}$$

Here X, Y are objects and g is a morphism of $\mathbf{A}$. Their F_{com}-images are sets and a map, respectively. While there can be non-injective monomorphisms in $\mathbf{A}$, this cannot happen in $F_{\mathrm{com}}(\mathbf{A})$. Thus, $F_{\mathrm{com}}(\mathbf{A})$ is *nice* in the sense that injective ⇔ mono. Furthermore, the vertical arrows are onto maps.

An illustration of the cometic functor

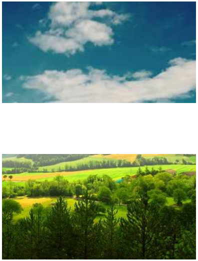
$$\begin{tikzcd}
X \arrow[r, "g"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
F_{\mathrm{com}}(X) \arrow[r, "F_{\mathrm{com}}(g)"'] \arrow[d, "\boldsymbol{\pi}_X^{\mathrm{com}}"] & F_{\mathrm{com}}(Y) \arrow[d, "\boldsymbol{\pi}_Y^{\mathrm{com}}"'] \\
\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(X) \arrow[r, "\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(g)"] & \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(Y)
\end{tikzcd}$$

Here X, Y are objects and g is a morphism of **A**. Their F_{com}-images are sets and a map, respectively. While there can be non-injective monomorphisms in **A**, this cannot happen in $F_{\mathrm{com}}(\mathbf{A})$. Thus, $F_{\mathrm{com}}(\mathbf{A})$ is *nice* in the sense that injective ⇔ mono. Furthermore, the vertical arrows are onto maps.

An illustration of the cometic functor

$$\begin{tikzcd}
X \arrow[r, "g"] & Y \\
F_{\mathrm{com}}(X) \arrow[r, "F_{\mathrm{com}}(g)"'] \arrow[d, "\boldsymbol{\pi}^{\mathrm{com}}_X"] & F_{\mathrm{com}}(Y) \arrow[d, "\boldsymbol{\pi}^{\mathrm{com}}_Y"'] \\
\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(X) \arrow[r, "\boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(g)"] & \boldsymbol{\iota}_{\mathbf{A},\mathbf{Set}}(Y)
\end{tikzcd}$$



Here X, Y are objects and g is a morphism of **A**. Their F_{com}-images are sets and a map, respectively. While there can be non-injective monomorphisms in **A**, this cannot happen in $F_{\mathrm{com}}(\mathbf{A})$. Thus, $F_{\mathrm{com}}(\mathbf{A})$ is *nice* in the sense that injective ⇔ mono. Furthermore, the vertical arrows are onto maps.

Thank you for your attention

These slides are available

`http://www.math.u-szeged.hu/~czedli/`

`http://www.math.u-szeged.hu/~czedli/conference-talks/conflectures0.html`