Free idempotent generated semigroups: subsemigroups, retracts and maximal subgroups

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(e, f) is **a basic pair** if

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4. Any biordered set E is $E(S)$ for some semigroup S **Easdown (1985)**.

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4. The morphism ϕ induces a bijection between the set of all $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) in the $\mathcal{D}$-class of $\bar{e}$ in $\mathrm{IG}(E)$ and the corresponding set in $S' = \langle E(S) \rangle$.

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Note that in the cases above if rank e is $n-1$ then $H_{\overline{e}}$ is free and if rank e is n or 0 then $H_{\overline{e}}$ is trivial.

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For $e \in E$, when $H_{\overline{e}}$ in $\mathrm{IG}(E)$ is isomorphic to $H_{\overline{e}}$ in $\mathrm{IG}(F)$?

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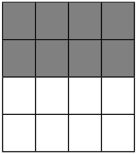

where the grey part denotes D_e^S of S.

Maximal subgroups of IG(E): semigroups and subsemigroups

Put $D = D_e^T$ and $D' = D_e^S = D_e^T \cap S$.

Let I index the $\mathcal{R}$-classes of D, and let $I' \subseteq I$ index the $\mathcal{R}$-classes of D'.

Let Λ index the $\mathcal{L}$-classes of D and D'.

R_i denotes the $\mathcal{R}$-class in D indexed by $i \in I$.

L_λ denotes the $\mathcal{L}$-class in D indexed by $\lambda \in \Lambda$.

$H_{i\lambda} = R_i \cap L_\lambda$; if $H_{i\lambda}$ is group, $e_{i\lambda}$ denotes its identity.

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Suppose that S and T satisfy Condition (R) and D is singularisable. We say in addition that S and T satisfy **Condition** (P) if for every $\mathcal{D}$-class $D = D_e^T$ of $e \in E$, we have that for all $i \in I$, there exists $i' \in I'$ such that for all $j \in I$ and $\lambda, \mu \in \Lambda$,

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Corollary The following idempotents are good.

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Lemma Let S, T, E, F be defined as above. Let $e \in E$ and put $D' = D_e^S$ and $D = D_e^T$. Suppose that (i) Condition (R) holds, (ii) D is stable and (iii) for each $f \in D \cap F$ we have $f \,\mathcal{L}^T\, f\theta$. Then Condition (P) holds.

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Proposition Let S, T, E, F be defined as above. Suppose that $e \in E$ and $D' = D_e^S$ is stable and singularisable via up-down singular squares. Then the maximal subgroup of $\overline{e}$ in $\mathrm{IG}(E)$ is isomorphic to the maximal subgroup of $\overline{e}$ in $\mathrm{IG}(F)$ and to the maximal subgroup of $\overline{e^*}$ in $\mathrm{IG}(F)$.

Thank you for listening!