$\mathscr{R}$-cross-sections of the semigroup of order-preserving transformations of a finite chain

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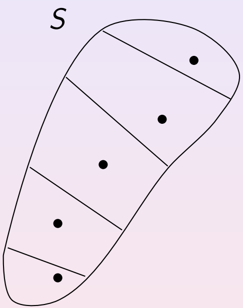
ρ-cross-section



ρ is an equivalence on S

transversal

If a transversal of ρ is a semigroup then it is called a ρ-cross-section.

Cross-sections of Green's relations in classical transformation semigroups

$[n] = \{1, 2, \ldots, n\}$,
$\mathcal{T}_n$ transformation semigroup (written on the left)

Green's relation $\mathcal{K}$	$\mathcal{H}$	$\mathcal{R}$	$\mathcal{J} = \mathcal{D}$	$\mathcal{L}$
$\mathcal{K}$-cross-sections	exist only for $n = 1, 2$, unique	exist, unique up to isomorphism	exist, no description is known	exist, not unique, even up to isomorphism

- Classical Finite Transformation Semigroups: An Introduction. (Ganyushkin O., Mazorchuk V., 2009)
- Bondar E. [2014, 2016]

Semigroup $\mathcal{O}_n$ of order-preserving transformations:

$$\alpha \in \mathcal{T}_n : \text{for all } x, y \in [n] \;\; x \le y \text{ implies } x\alpha \le y\alpha.$$

Green's relations of $\mathcal{O}_n$ are just the restrictions of the corresponding Green's relations on $\mathcal{T}_n$:
$\forall \alpha, \beta \in \mathcal{O}_n$

a) $\alpha \mathscr{R} \beta$ if and only if $\ker(\alpha) = \ker(\beta)$;
b) $\alpha \mathscr{L} \beta$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$.

$\mathscr{L}$-cross-sections of $\mathcal{O}_n$

The description of $\mathscr{L}$-cross-sections of $\mathcal{O}_n$ follows from the description of $\mathscr{L}$-cross-sections for $\mathcal{T}_n$.

Semigroup of order-preserving transformations

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Higgins' embedding

$\mathcal{O}_n$ can be embedded in dual $\mathcal{O}^*_{n+1}$ (P. Higgins, 1995)

$K = \{k_1, k_2, \ldots, k_t\}$ is the set, written in ascending order, of the maximum members of its kernel classes

$\mathrm{im}(\alpha) = \{r_1, r_2, \ldots, r_t\}$,

$k_i\alpha = r_i$ for all $1 \le i \le t$.

$$x\alpha^* = \begin{cases} 1 & \text{if } x \le r_1, \\ k_i + 1 & \text{if } r_i < x < r_{i+1},\ 1 \le i \le t \end{cases}$$

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1 •

3 •

5 •

. . .

•
$2n+1$

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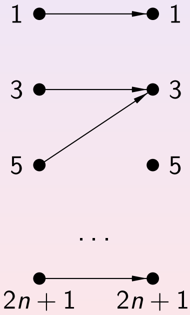

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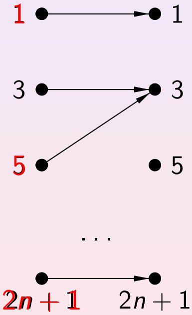

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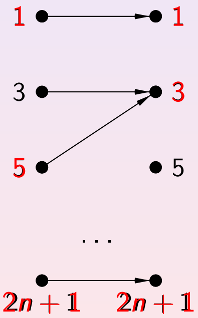

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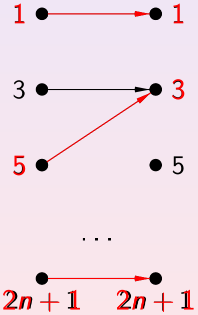

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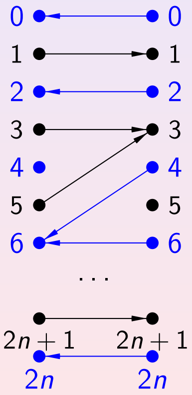

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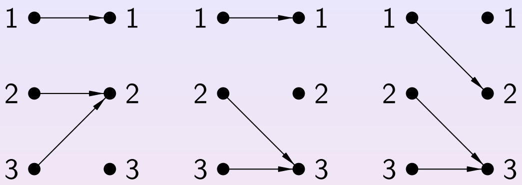
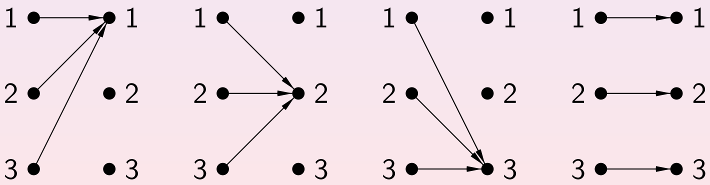
$\mathscr{L}$-cross-section of $\mathcal{O}_3$ and its dual





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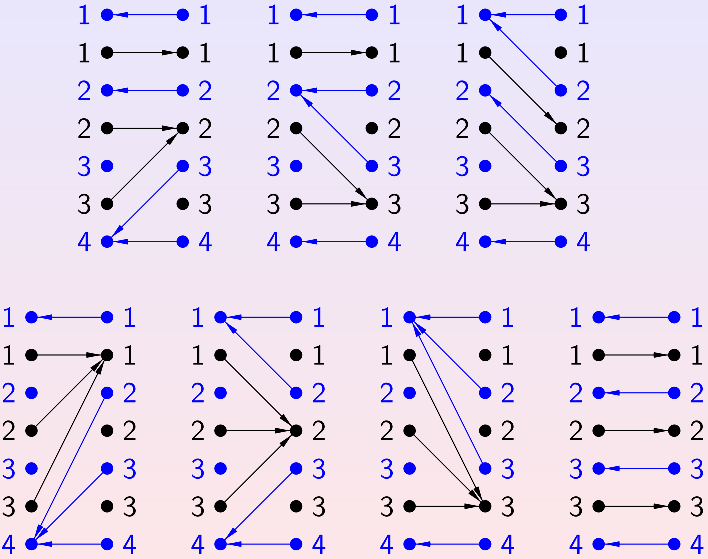

A homomorphism $\Gamma_1 \to \Gamma_2$: sends the root to the root, preserves the parent–child relation and the genders.
Γ_1 subordinates Γ_2 if there exists a 1-1 homomorphism $\Gamma_1 \to \Gamma_2$.

Respectful trees

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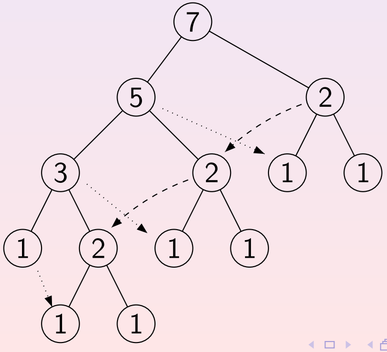
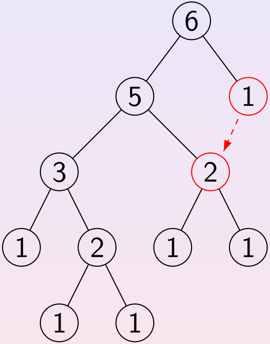
A respectful binary tree is a full binary tree such that conditions:

(S1) if a male vertex has a nephew, the nephew subordinates his uncle;

(S2) if a female vertex has a niece, the niece subordinates her aunt.



A full binary tree which is not respectful



Order-preserving trees

$\mathbf{r}$ denotes the root,
$\mathbf{s}(v)$ the son of a vertex v,
$\mathbf{d}(v)$ the daughter of a vertex v,
$\mathbf{p}(v)$ denotes the parent of v

We say a binary tree $T(n)$ is order-preserving for $([n], \leq)$, if the following conditions hold true:
1) if the root has the son or the daughter then

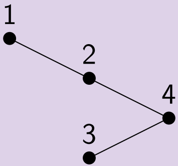
$$1 \leq \mathbf{s}(\mathbf{r}) < \mathbf{r} \quad \text{and} \quad \mathbf{r} < \mathbf{d}(\mathbf{r}) \leq n \quad \text{respectively.}$$

2) if $v \in T(n)$ is a vertex and for $\mathbf{p}(v)$ and some $x, y \in [n]$ the condition $x \leq \mathbf{p}(v) \leq y$ holds, then

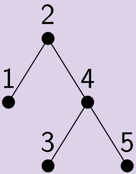
$$\begin{cases} x \leq v < \mathbf{p}(v), & \text{if } v \text{ is the son } \mathbf{p}(v), \\ \mathbf{p}(v) < v \leq y, & \text{if } v \text{ is the daughter } \mathbf{p}(v). \end{cases}$$

Order-preserving trees

$T(4)$



$T_1(5)$



$T_2(5)$

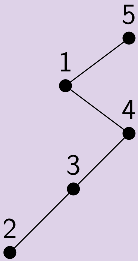
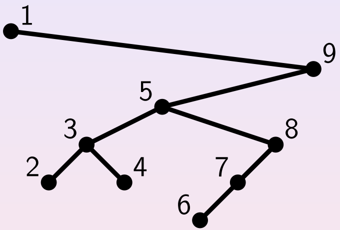


Diagram presentation of $([n], \prec)$



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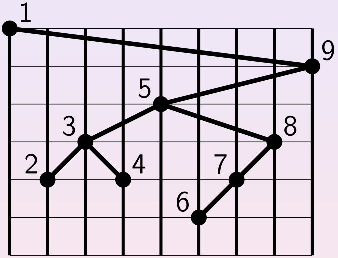

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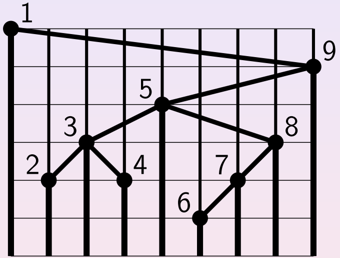

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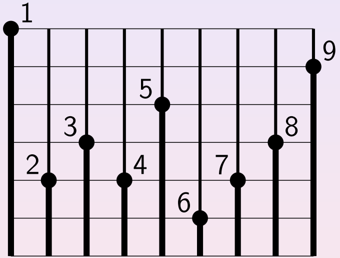
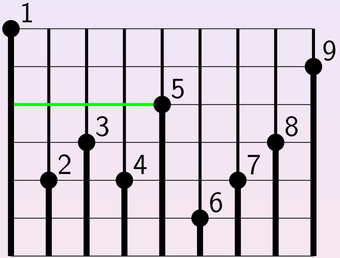


«Inner» trees



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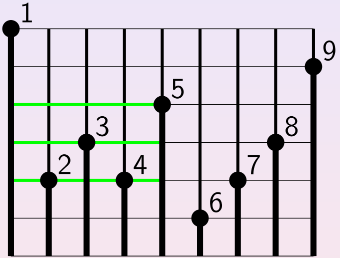
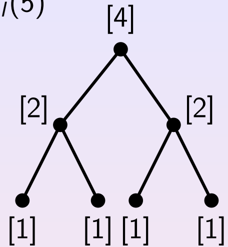


$\Gamma_I(5)$



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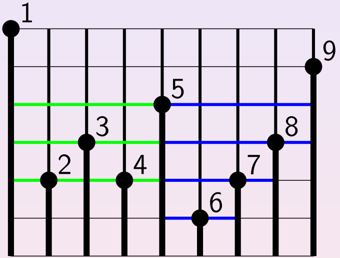
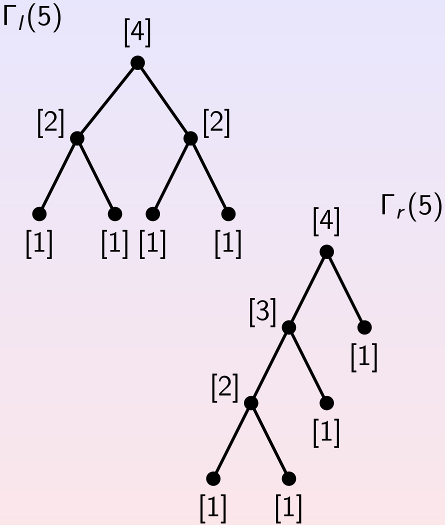
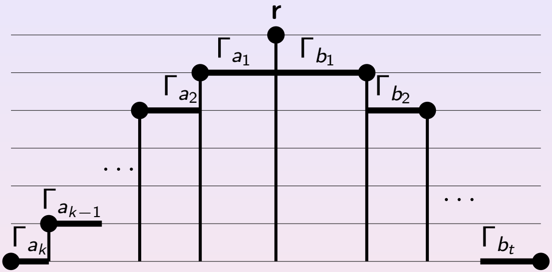




Sketch of an order-preserving tree for an $\mathscr{R}$-cross-section of $\mathcal{O}_n$



Γ_{a_i} (Γ_{b_j}) respectful trees on a_i- (b_j-)element set respectively, each tree subordinates the tree above

$a_1 + a_2 + \ldots + a_k = \mathbf{r} - 1$, $a_k \leq a_{k-1} \geq \ldots \geq a_1$,

$b_1 + b_2 + \ldots + b_t = n - \mathbf{r}$, $b_1 \leq b_2 \leq \ldots \leq b_t$

$([n], \prec)$ an order-preserving tree
$\widetilde{K_m}$ a partition of $[n]$ into m convex intervals
Order-preserving tree $T(\widetilde{K_m})$ of intervals $\widetilde{K_m}$
$\varphi_{\prec}^{\widetilde{K_m}}$ a 1-1 homomorphism between the tree of partitions and $([n], \prec)$.
$\Phi_{\prec}$ a set of $\varphi_{\prec}^{\widetilde{K_m}}$, where $\widetilde{K_m}$ goes through all possible convex partitions of $[n]$

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Theorem

Given an order-preserving binary tree $([n], \prec)$ the set $\Phi_{\prec}$ constitutes an $\mathscr{R}$-cross-section of $\mathcal{O}_n$. Conversely, every $\mathscr{R}$-cross-section of $\mathcal{O}_n$ is isomorphic to $\Phi_{\prec}$ for an order-preserving binary tree $([n], \prec)$.

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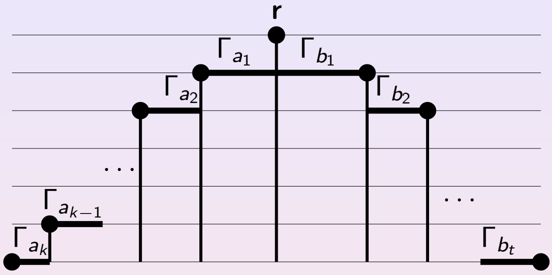

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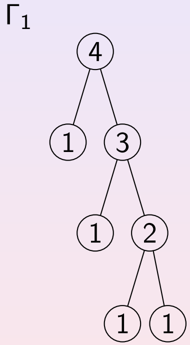
Similar respectful trees ($\Gamma_1 \sim \Gamma_2$)



Theorem

Let R_1, R_2 be two $\mathscr{R}$-cross-sections of $\mathcal{O}_n$.
$R_1 \cong R_2$ iff one of the following conditions holds

(1) *the diagram of R_1 is a mirror reflection of the diagram of R_2;*

(2) $\Gamma_{a_i} \sim \Gamma'_{a_i}$ *for some* $1 \leq i \leq k$*, or* $\Gamma_{b_j} \sim \Gamma'_{b_j}$ *for some* $1 \leq j \leq t$*, while other components are the same.*