Divisibility of ultrafilters

Boris Šobot

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AAA94 / NSAC2017

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Divisibility of ultrafilters

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Image: A matrix

${\cal N}$ - discrete topological space on the set of natural numbers

Ultrafilter: nonempty $x \subseteq P(N)$ such that: (1) $A, B \in x \Rightarrow A \cap B \in x;$ (2) $A \in x, A \subseteq B \Rightarrow B \in x;$ (3) $A \subseteq N \Rightarrow A \in x \lor A^c \in x.$

If $A \in x$, we say: ultrafilter x concentrates on A.

 βN - the set of ultrafilters on N

Principal ultrafilters $\{A \subseteq N : n \in A\}$ are identified with respective elements $n \in N$

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Algebra in βN

Every function $f:N\to N$ can be extended uniquely to continuous $\tilde{f}:\beta N\to\beta N$

The multiplication can be extended to βN as follows:

 $A \in p \cdot q \Leftrightarrow \{n \in N : A/n \in q\} \in p.$

where, for $A \subseteq N$ and $n \in N$:

$$A/n = \{m \in N : mn \in A\} = \left\{\frac{a}{n} : a \in A, n \mid a\right\}$$

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$\mathcal{U} = \{ S \subseteq N : S \text{ is upward closed for } | \}$

 $\mathcal{V} = \{ S \subseteq N : S \text{ is downward closed for } | \}$

 $p \,\widetilde{\mid}\, q \text{ iff } p \cap \mathcal{U} \subseteq q \text{ iff } q \cap \mathcal{V} \subseteq p$

The restriction of $\tilde{\mid}$ to N^2 is the usual \mid $\tilde{\mid}$ is reflexive and transitive, but not antisymmetric



 $\begin{aligned} \mathcal{U} &= \{S \subseteq N : S \text{ is upward closed for } \mid \} \\ \mathcal{V} &= \{S \subseteq N : S \text{ is downward closed for } \mid \} \\ &\qquad p \mid q \text{ iff } p \cap \mathcal{U} \subseteq q \text{ iff } q \cap \mathcal{V} \subseteq p \end{aligned}$ The restriction of $\mid \text{ to } N^2 \text{ is the usual } \mid \\ &\qquad \mid \text{ is reflexive and transitive, but not antisymmetric} \end{aligned}$

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Prime ultrafilters: $p\in\beta N\setminus\{1\}$ divisible only by 1 and themselves

 $p \in \beta N$ is prime iff $P \in p$ (*P* - the set of prime numbers)

So there are $2^{\mathfrak{c}}$ prime ultrafilters

For every $x \in \beta N \setminus \{1\}$ there is prime p such that $p \mid x$

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Prime ultrafilters



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$$A^2 = \{a^2 : a \in A\}$$

The only ultrafilter above p containing P^2 is

 p^2 is generated by $\{A^2 : A \in p\}$

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$A^{(2)} = \{ab : a, b \in A, GCD(a, b) = 1\}$

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$$A^{(2)} = \{ab : a, b \in A, GCD(a, b) = 1\}$$
$$F_{(p,2)} = \{A^{(2)} : A \in p, A \subseteq P\}$$

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Ultrafilters containing $F_{(p,2)}$ are also divisible only by 1, p and themselves

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Example. $p \cdot p \supseteq F_{(p,2)}$

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Example. $p \cdot p \supseteq F_{(p,2)}$

There are either finitely many or $2^{\mathfrak{c}}$ ultrafilters containing $F_{(p,2)}$

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$AB = \{ab: a \in A, b \in B, GCD(a, b) = 1\}$

$$F_{(p,1),(q,1)} = \{AB : A \in p, B \in q, A, B \subseteq P \text{ are disjoint}\}\$$

Ultrafilters containing $F_{(p,1),(q,1)}$ are divisible only by 1, p, q and themselves

They are exactly ultrafilters containing AB for some disjoint $A, B \subseteq P$

Example. $p \cdot q, q \cdot p \supseteq F_{(p,1),(q,1)}$

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 $n \in N$ is perfect if $\sigma(n) = 2n$ $n \in N$ is multiperfect if $\exists k \in N \sigma(n) = kn$

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$$n \in N$$
 is perfect if $\sigma(n) = 2n$
 $n \in N$ is multiperfect if $\exists k \in N \ \sigma(n) = kn$

Is $N_P = \{n \in N : n \text{ is perfect}\}$ infinite? Is $N_M = \{n \in N : n \text{ is multiperfect}\}$ infinite?

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$$|N_M| = \aleph_0 \qquad |N_P| = \aleph_0$$
$$\exists x \in \beta N \ x \ \widetilde{\mid} \ \widetilde{\sigma}(x) \quad \exists x \in \beta N \ \widetilde{\sigma}(x) = 2x$$

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$$\begin{split} |N_M| &= \aleph_0 & |N_P| &= \aleph_0 \\ & \updownarrow \\ \exists x \in \beta N \ x \ \widetilde{\mid} \ \widetilde{\sigma}(x) & \exists x \in \beta N \ \widetilde{\sigma}(x) = 2x \end{split}$$

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W. W. Comfort, S. Negrepontis: The theory of ultrafilters
R. C. Walker: The Stone-Čech compactification
N. Hindman, D. Strauss: Algebra in the Stone-Čech compactification, theory and applications
B. Šobot: Divisibility in the Stone-Čech compactification, Rep. Math. Logic 50 (2015), 53-66.
B. Šobot: Divisibility orders in the Stone-Čech compactification, submitted

[6] B. Šobot: -*divisibility of ultrafilters*, submitted

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Thank you for your attention!



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