Divisibility of ultrafilters

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The Stone-Čech compactification

N - discrete topological space on the set of natural numbers

Ultrafilter: nonempty $x \subseteq P(N)$ such that:
(1) $A, B \in x \Rightarrow A \cap B \in x$;
(2) $A \in x, A \subseteq B \Rightarrow B \in x$;
(3) $A \subseteq N \Rightarrow A \in x \veebar A^c \in x$.

If $A \in x$, we say: ultrafilter x concentrates on A.

βN - the set of ultrafilters on N

Principal ultrafilters $\{A \subseteq N : n \in A\}$ are identified with respective elements $n \in N$

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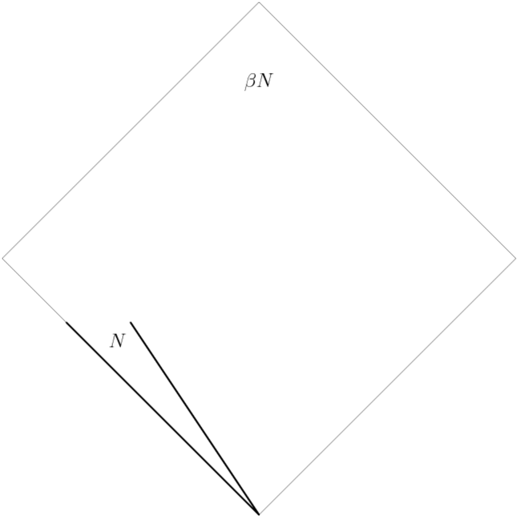

Algebra in βN

Every function $f : N \to N$ can be extended uniquely to continuous $\tilde{f} : \beta N \to \beta N$

The multiplication can be extended to βN as follows:

$$A \in p \cdot q \Leftrightarrow \{n \in N : A/n \in q\} \in p.$$

where, for $A \subseteq N$ and $n \in N$:

$$A/n = \{m \in N : mn \in A\} = \left\{\frac{a}{n} : a \in A, n \mid a\right\}$$

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$\widetilde{|}$-divisibility

$\mathcal{U} = \{S \subseteq N : S \text{ is upward closed for } |\}$

$\mathcal{V} = \{S \subseteq N : S \text{ is downward closed for } |\}$

$$p \,\widetilde{|}\, q \text{ iff } p \cap \mathcal{U} \subseteq q \text{ iff } q \cap \mathcal{V} \subseteq p$$

The restriction of $\widetilde{|}$ to N^2 is the usual $|$

$\widetilde{|}$ is reflexive and transitive, but not antisymmetric

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Prime ultrafilters

Prime ultrafilters: $p \in \beta N \setminus \{1\}$ divisible only by 1 and themselves

$p \in \beta N$ is prime iff $P \in p$
(P - the set of prime numbers)

So there are $2^{\mathfrak{c}}$ prime ultrafilters

For every $x \in \beta N \setminus \{1\}$ there is prime p such that $p \,\widetilde{\mid}\, x$

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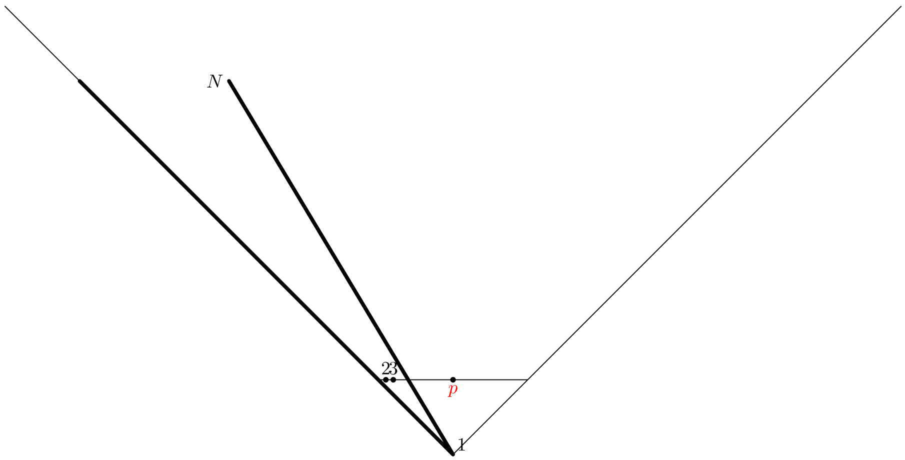

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$$A^2 = \{a^2 : a \in A\}$$

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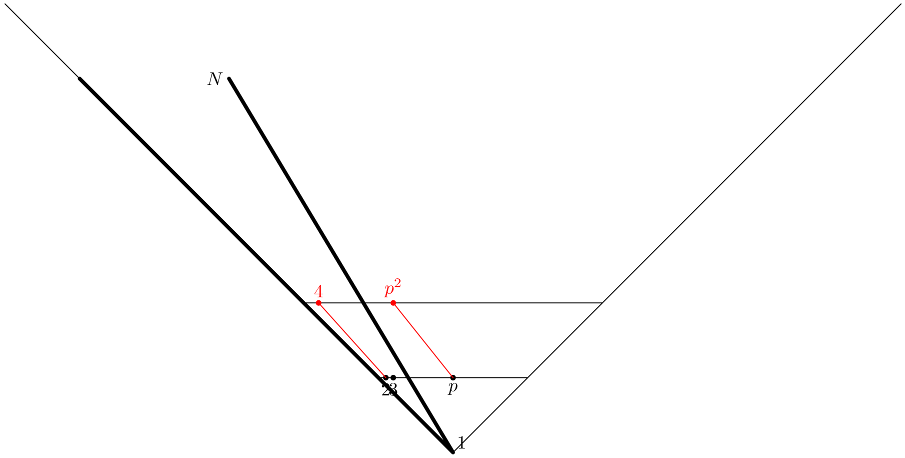

The second level

$$A^{(2)} = \{ab : a, b \in A, GCD(a, b) = 1\}$$

$$F_{(p,2)} = \{A^{(2)} : A \in p, A \subseteq P\}$$

Ultrafilters containing $F_{(p,2)}$ are also divisible only by 1, p and themselves

Example. $p \cdot p \supseteq F_{(p,2)}$

There are either finitely many or $2^{\mathfrak{c}}$ ultrafilters containing $F_{(p,2)}$

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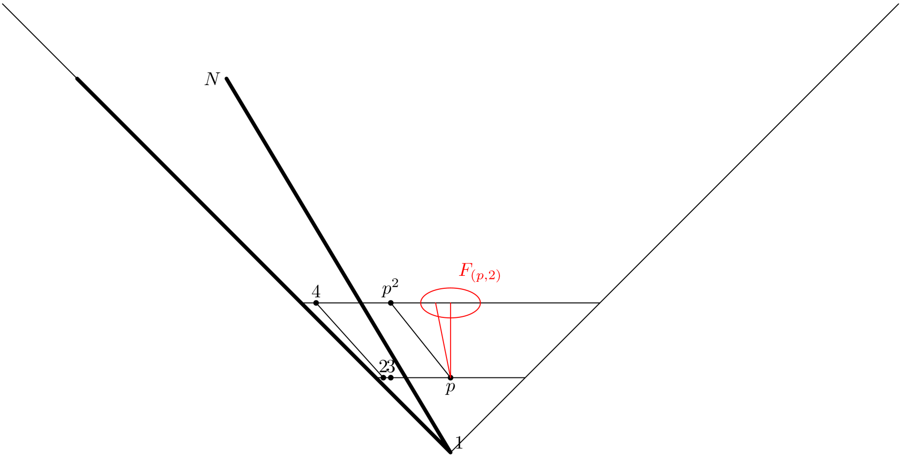

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$$AB = \{ab : a \in A, b \in B, GCD(a,b) = 1\}$$

$$F_{(p,1),(q,1)} = \{AB : A \in p, B \in q, A, B \subseteq P \text{ are disjoint}\}$$

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They are exactly ultrafilters containing AB for some disjoint $A, B \subseteq P$

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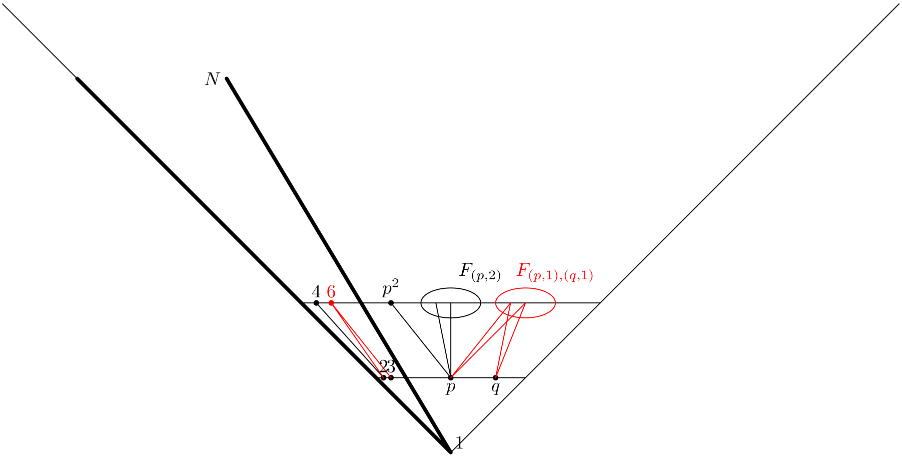

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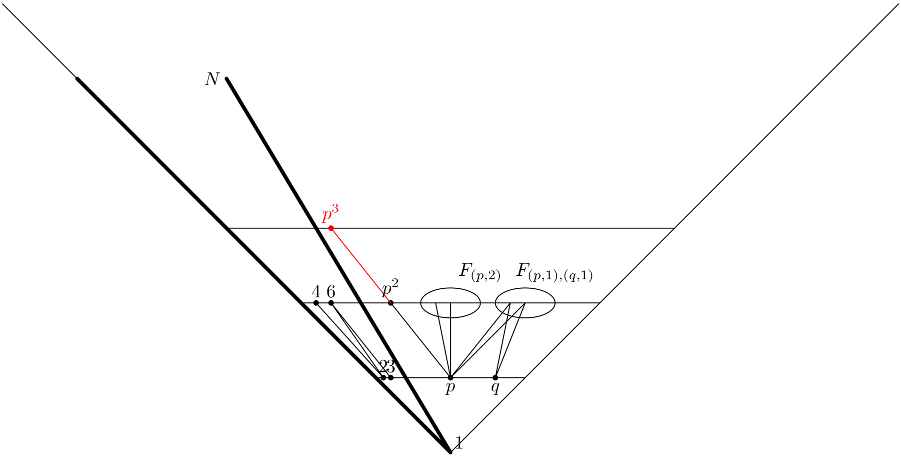

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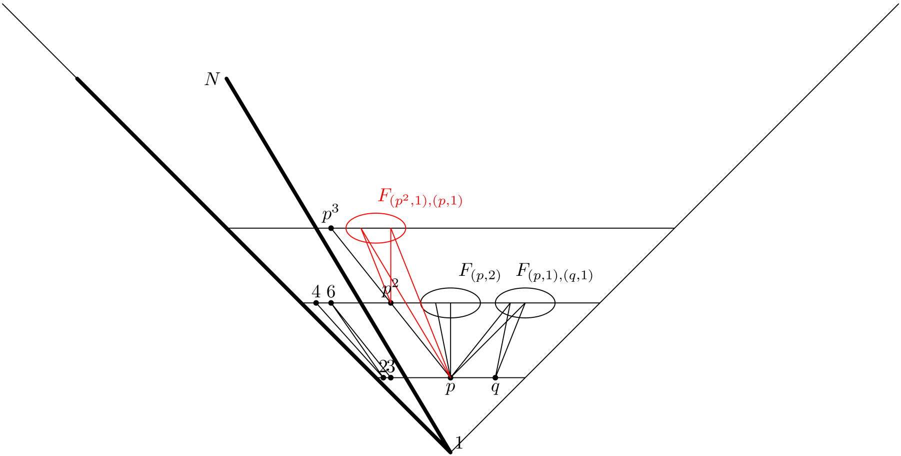

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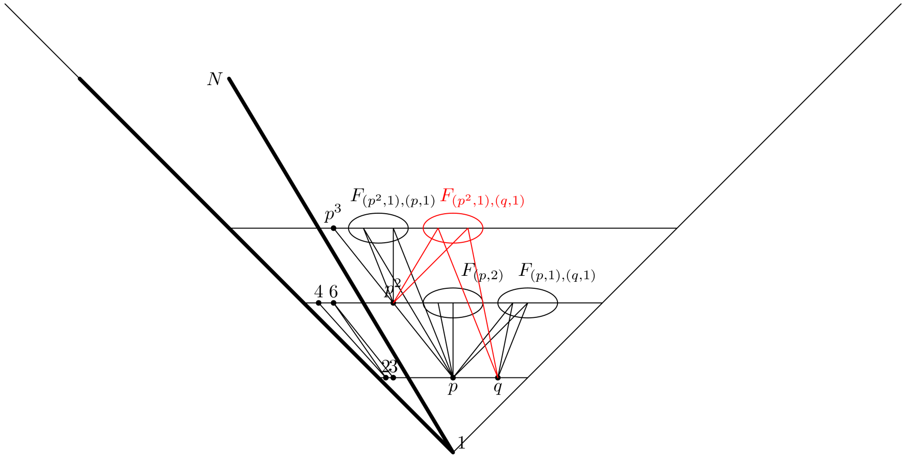

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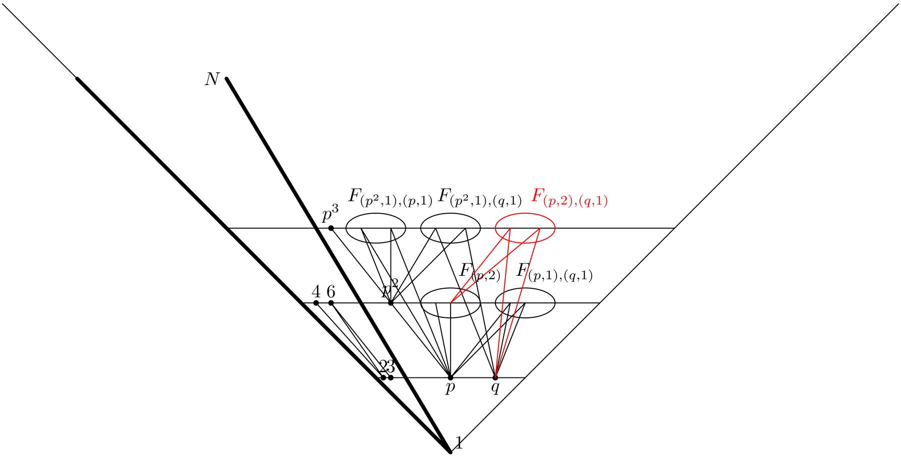

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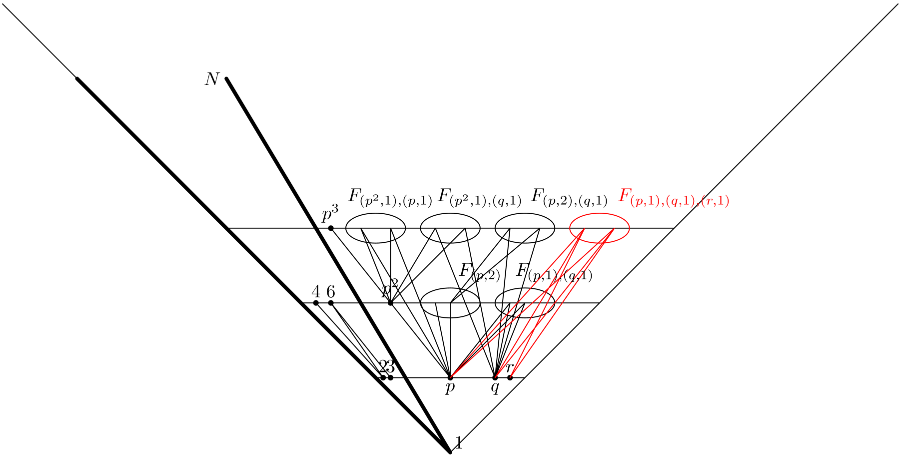

Example: perfect numbers

$n \in N$ is perfect if $\sigma(n) = 2n$
$n \in N$ is multiperfect if $\exists k \in N\ \sigma(n) = kn$

Is $N_P = \{n \in N : n \text{ is perfect}\}$ infinite?
Is $N_M = \{n \in N : n \text{ is multiperfect}\}$ infinite?

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$$|N_M| = \aleph_0 \qquad\qquad |N_P| = \aleph_0$$

$$\exists x \in \beta N\ x \mathrel{\widetilde{\mid}} \widetilde{\sigma}(x) \quad \exists x \in \beta N\ \widetilde{\sigma}(x) = 2x$$

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References

[1] W. W. Comfort, S. Negrepontis: *The theory of ultrafilters*
[2] R. C. Walker: *The Stone-Čech compactification*
[3] N. Hindman, D. Strauss: *Algebra in the Stone-Čech compactification, theory and applications*
[4] B. Šobot: *Divisibility in the Stone-Čech compactification*, Rep. Math. Logic 50 (2015), 53-66.
[5] B. Šobot: *Divisibility orders in the Stone-Čech compactification*, submitted
[6] B. Šobot: $\widetilde{\mid}$*-divisibility of ultrafilters*, submitted

Thank you for your attention!