Reflection-closed varieties of multisorted algebras and minor identities I

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Outline

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"historical" remarks

Notions and notations

Results

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background 2015:

L. Barto, J. Opršal, M. Pinsker, *The wonderland of reflections*, arXiv: 1510.04521 (first version Oct. 2015), as tool for CSP

reflections of operations $f : A^n \to A$

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- J. Opršal, R.P.: the appropriate tool for characterization of reflections: pairs of relations (instead of relations)
- E. Lehtonen observed the connection to the paper N. Pippenger, *Galois theory for minors of finite functions*, Discrete Math. 254 (2002), 405–419.

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multi-sorted stuff

S set of *sorts*

S-sorted set: $A := (A_s)_{s \in S}$ $\qquad$ $S_A := \{s \in S \mid A_s \neq \emptyset\}$

S-sorted mapping: $h : A \to B$ where $h = (h_s)_{s \in S}$, $h_s : A_s \to B_s$

S-sorted algebra: $\mathbf{A} := (A, (f_i)_{i \in I})$ ($A = (A_s)_{s \in S}$ S-sorted set)

fundamental operations are of the form

$$f : \underbrace{A_{s_1} \times \ldots \times A_{s_n}}_{=:Aw} \to A_s$$

i.e., $f : A_w \to A_s$, where $w := s_1 \ldots s_n \in W(S)$, $s \in S$

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Reflections

For algebras $\mathbf{A} = (A, f)$ and $\mathbf{A}' = (A', f')$,

$\mathbf{A}'$ (or f') is a *reflection* of $\mathbf{A}$ (or f, resp.)

$$\begin{tikzcd} {} \arrow[r, "f"] & {} \arrow[d] \\ {} \arrow[u] \arrow[r, "f'"] & {} \end{tikzcd}$$

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$:\iff \exists h : A' \to A \; \exists h' : A \to A'$

$$
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
h \uparrow & & \downarrow h' \\
A' & \xrightarrow{f'} & A'
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$$\begin{tikzcd} A^n \arrow[r, "f"] & A \arrow[d, "h'"] \\ A'^n \arrow[u, "(h)^n"] \arrow[r, "f'"] & A' \end{tikzcd}$$

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For 2-sorted algebras $\mathbf{A} = (A_1, A_2, f)$ and $\mathbf{A}' = (A_1', A_2', f')$,

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$:\iff \exists h_1 : A_1' \to A_1 \; \exists h_2' : A_2 \to A_2'$

$$\begin{tikzcd}
A_1^n \arrow[r, "f"] & A_2 \arrow[d, "h_2'"] \\
{A_1'}^n \arrow[u, "(h_1)^n"] \arrow[r, "f'"] & A_2'
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For multi-sorted algebras $\mathbf{A} = ((A_s)_{s\in S}, f)$ and $\mathbf{A}' = ((A'_s)_{s\in S}, f')$

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$:\iff \forall\, s \in S \setminus \{s \mid A'_s = \emptyset\}\ \exists h_s : A'_s \to A_s\ \exists h'_s : A_s \to A'_s$
i.e. $\exists\ S_{A'}$-sorted mappings $h : A' \to A$, $h' : A \to A'$

$$\begin{tikzcd} A_w \arrow[r, "f"] & A_s \arrow[d, "h'_s"] \\ A'_w \arrow[u, "h_w"] \arrow[r, "f'"] & A'_s \end{tikzcd}$$

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Remark: subalgebras and homomorphic images are special cases

(multisorted similarity) Types (of algebras)

type $\tau = (S, \Sigma, \mathrm{dec})$
S set of *sorts*
Σ set of *operation symbols*
$\mathrm{dec} : \Sigma \to W(S) \times S$ *declaration* for operation symbols
$\mathrm{dec}(f) = (w, s)$, $w = s_1 \dots s_n \in W(S)$, $s \in S$

$w = s_1 \dots s_n =$ *arity* $\mathrm{ar}(f)$ of f,
$s_1, \dots, s_n$ *input sorts*, s *(output) sort* of f.

interpretation of the symbol f in a (multisorted) algebra $\mathbf{A}$:
$f^{\mathbf{A}} : A_w \to A_s$, i.e., $f^{\mathbf{A}} : A_{s_1} \times \dots \times A_{s_n} \to A_s$

Remark: such functions can exist only if the declaration for f is *reasonable* in the multi-sorted set A, i.e., if $A_s = \emptyset$ then $A_{s_i} = \emptyset$ for at least one $i \in \{1, \dots, n\}$.

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Minor terms and minor identities

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A *minor term* (or term of height 1, or h1-term) *of sort s of type τ over Y* is a term t of the form

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standard variables: $X = (X_s)_{s\in S}$ where $X_s := \{x^s_1, x^s_2, x^s_3 \dots\}$ (all X_s pairwise disjoint)

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if for every valuation map $\beta : X_{|S'} \to A$ we have $\beta^\#(t_1) = \beta^\#(t_2)$
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$\models$ induces the following Galois connection ($\mathcal{K} \subseteq \mathrm{Alg}(\tau)$, $\mathcal{J} \subseteq MID_\tau(X)$):

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Outline

"historical" remarks

Notions and notations

Results

Reflection-closed varieties (RP-varieties)

recall variety generated by $\mathcal{K} \subseteq \mathrm{Alg}(\tau)$: $\mathsf{HSP}\mathcal{K}$
now in addition reflection closure: reflection-closed varieties can be characterized as $\mathsf{RHSP}\mathcal{K} = \mathsf{RP}\mathcal{K}$
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i.e., minor-equational classes = RP-varieties.

For usual (one-sorted) algebras this was proved in the wonderland paper by Barto, Opršal, Pinsker

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Sketch of the proof (Part 1)

(straightforward) Part I: $\mathrm{RP}\mathcal{K} \subseteq \mathrm{Mod}\,\mathrm{mId}\,\mathcal{K}$

or equivalently: $\mathrm{RP}(\mathrm{Mod}\,\mathcal{J}) \subseteq \mathrm{Mod}\,\mathcal{J}$ for any set $\mathcal{J}$ of identities.

$\mathrm{P}(\mathrm{Mod}\,\mathcal{J}) \subseteq \mathrm{Mod}\,\mathcal{J}$ holds by the classical Birkhoff HSP-theorem.

It remains to show $\mathrm{R}(\mathrm{Mod}\,\mathcal{J}) \subseteq \mathrm{Mod}\,\mathcal{J}$.

Let $\mathbf{B} \in \mathrm{R}(\mathrm{Mod}\,\mathcal{J})$; then $\mathbf{B}$ is an (h, h')-reflection of some $\mathbf{A} \in \mathrm{Mod}\,\mathcal{J}$ for some $h\colon B \to A$ and $h'\colon A \to B$.
We need to show that $\mathbf{B} \models (S', f_\sigma, g_\pi)$ for every $(S', f_\sigma, g_\pi) \in \mathcal{J}$.
Let $\beta\colon X_{S'} \to B$ be a valuation. Then

$$\begin{aligned}\beta^{\#}(f_\sigma) &= f^{\mathbf{B}}(\beta \circ \sigma) = h'(f^{\mathbf{A}}(h \circ \beta \circ \sigma)) = h'(f_\sigma^{\mathbf{A}}(h \circ \beta)) \\ &= h'(g_\pi^{\mathbf{A}}(h \circ \beta)) = h'(g^{\mathbf{A}}(h \circ \beta \circ \pi)) = g^{\mathbf{B}}(\beta \circ \pi) = \beta^{\#}(g_\pi),\end{aligned}$$

where the equality $=$ holds because $\mathbf{A} \models (S', f_\sigma, g_\pi)$, whence $f_\sigma^{\mathbf{A}}(h \circ \beta) = (h \circ \beta)^{\#}(f_\sigma) = (h \circ \beta)^{\#}(g_\pi) = g_\pi^{\mathbf{A}}(h \circ \beta)$.

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(straightforward) Part I: $\mathrm{RP}\mathcal{K} \subseteq \mathrm{Mod}\,\mathrm{mId}\,\mathcal{K}$
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(crucial) Part 2: $\mathrm{Mod}\,\mathrm{mId}\,\mathcal{K} \subseteq \mathrm{RP}\mathcal{K}$ (type $\tau = (S, \Sigma, \mathrm{dec})$)

Let $\mathbf{B} \in \mathrm{Mod}\,\mathrm{mId}\,\mathcal{K}$ (to show $\mathbf{B} \in \mathrm{RP}\mathcal{K}$) (w.l.o.g. all B_s disjoint)
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$$\mathcal{N} := \{(S_Y, t_1, t_2) \in MID_\tau(Y) \mid \mathcal{K} \not\models (S_Y, t_1, t_2)\}$$

(recall: $S_Y = \{s \in S \mid Y_s \neq \emptyset\}$).
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Take $\mathbf{P} := \prod_{\nu \in \mathcal{N}} \mathbf{A}_\nu$ (product of all the counterexamples).
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Finally, one can show that $\mathbf{B}$ is a reflection of $\mathbf{P}$.

Consequently, $\mathbf{B} \in \mathrm{RP}\mathcal{K}$. $(\square)$

more in detail: $\mathbf{B}$ is a (h, h')-reflection of $\mathbf{P}$:

$h = (h_s)_{s \in S_Y}$ such that $h_s \colon B_s \to P_s$ is the map $b \mapsto \overline{b}$, where for $y := b$ we define $\overline{y} := (\beta_\nu(y))_{\nu \in \mathcal{N}}$ (β_ν is the valuation witnessing $\mathbf{A}_\nu \not\models \nu$).

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The case $\mathcal{N} = \emptyset$ needs an extra consideration (then $\mathbf{B}$ is a reflection of the trivial algebra $\mathbf{D}$ with $D_s = \{\emptyset\}$ one-element set for each $s \in S$, and we have $\mathbf{D} = \prod \emptyset$).

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$h = (h_s)_{s \in S_Y}$ such that $h_s \colon B_s \to P_s$ is the map $b \mapsto \overline{b}$, where for $y := b$ we define $\overline{y} := (\beta_\nu(y))_{\nu \in \mathcal{N}}$ (β_ν is the valuation witnessing $\mathbf{A}_\nu \not\models \nu$).
$h' = (h'_s)_{s \in S_B}$ such that $h'_s \colon P_s \to B_s$ is the following map:

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i.e., $h'_s(f^{\mathbf{P}}(\overline{y}_1, \ldots, \overline{y}_n)) = f^{\mathbf{B}}(b_1, \ldots, b_n) = f^{\mathbf{B}}(h_w(y_1, \ldots, y_n))$ according to the reflection property.
The case $\mathcal{N} = \emptyset$ needs an extra consideration (then $\mathbf{B}$ is a reflection of the trivial algebra $\mathbf{D}$ with $D_s = \{\emptyset\}$ one-element set for each $s \in S$, and we have $\mathbf{D} = \prod \emptyset$).

Sketch of the proof (Part 2), continued

Finally, one can show that **B** is a reflection of **P**.

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Thank you for your ATTENTION!